

The Ant and the Honey

Shorter Paths on Regular Prisms and on the Cylinder

Zlatko Sirotić

Abstract

An ant sits on one base of a regular n -prism and a drop of honey on the opposite base, both on the bisector of their base and at equal distance from the rim. The “obvious” path — across one base, up a lateral face, across the other base — has length $D = h + b$, independent of where along the bisector the two points sit. We ask when a shorter path exists, how many there are, how long they are, and at what angle the ant must set off.

The problem is solved completely for every n . A single pair of formulas gives the length and the direction of every candidate path on every regular prism; the cylinder follows as the limit $n \rightarrow \infty$. Along the way three phenomena appear that are easy to miss when prisms are treated one at a time: some paths exist only when the ant is far enough from the rim, some prisms admit no shorter path at all unless they are tall enough, and the number of shorter paths behaves very differently depending on whether helical paths are allowed.

Three conjectures are stated in Section 10 and settled in the appendices. The first is proved; the second turns out to be false except at one critical value; the third is confirmed, together with an asymptotic growth law.

Contents

1	The problem	3
1.1	Origin	3
1.2	Setting and notation	3
1.3	The two points and the obvious path	4
1.4	Method: unfolding and rolling	4
2	The square prism: the original puzzle	5
3	The square prism in general	5
3.1	When does a shorter path exist at all?	6
3.2	Two competing paths	6
3.3	How far from the rim can the ant be?	6
3.4	The most efficient shape	8
3.5	At what angle should the ant set off?	8
3.6	Summary for $2 \leq k \leq 5$	8
4	The triangular prism	8
4.1	Four classes	9
4.2	Which is shortest	9
4.3	A fifth class	9
4.4	Angles	11
4.5	Summary for $2 \leq k \leq 5$	11

5	The hexagonal prism	11
5.1	Four classes	11
5.2	Paths that do not exist for small p	12
5.3	Angles	14
6	The pentagonal prism	14
6.1	The eight classes	14
6.2	Angles	16
6.3	Thresholds and ranges	17
6.4	The graph	17
7	The cylinder	18
7.1	The limit of the general formula	18
7.2	Where the symmetry comes from	19
7.3	A closed parametric form	19
7.4	The case $p = 0$	19
7.5	Numerical values	21
8	One formula for every prism	21
8.1	Components	21
8.2	Angles	22
8.3	Feasibility	22
8.4	Thresholds in p	23
8.5	Odd and even prisms	23
9	When is there no shorter path at all?	24
9.1	Odd prisms	24
9.2	Even prisms	24
9.3	Thresholds in p : which prisms are affected	25
10	Three conjectures	26
A	Proof of H1	27
A.1	The criterion for p -independence	27
A.2	The length of a p -independent path	28
A.3	Even prisms	28
A.4	Odd prisms: existence	28
A.5	Classification of all p -independent paths	29
A.6	Which classes are shorter than D	30
A.7	Uniqueness	31
A.8	The main result	32
B	H2: refuted, except at one point	33
B.1	The numerical picture	33
B.2	Why it saturates	33
B.3	Why $k = 1$ is different	34
B.4	Conclusion	34
C	H3: confirmed, with a growth law	34
C.1	Structure	34
C.2	How many are shorter than the obvious path	35
C.3	Feasibility as $k \rightarrow \infty$	35
C.4	The growth law	35

C.5	Measurements	36
C.6	Conclusion	36

1 The problem

1.1 Origin

In 1981 a friend, Korado Korlević, posed the following puzzle. A rectangular box measures $4 \times 4 \times 10$. An ant sits on one 4×4 face, in the middle horizontally and 1 unit from the top edge; a drop of honey sits on the opposite face, in the middle horizontally and 1 unit from the bottom edge. The ant walks only on the surface. The obvious route — straight to the edge, along the box, then straight to the honey — has length $4 + 10 = 14$. Is there anything shorter?

There is, and it is found in a few hours with a sheet of paper. What took longer was the question that came next: *what if the box were a cylinder?* That one resisted, and was only settled in 2013. Returning afterwards to the prism, it became clear that the simpler problem had more in it than the original puzzle suggested — several distinct shorter routes rather than one, their number depending on the shape of the prism in a way that is not at all obvious, and a sharp difference between prisms with an odd and an even number of sides.

This text collects what is now known. The prisms are treated in the order in which they were originally studied — square, triangular, hexagonal, pentagonal — and then the cylinder. Section 8 gives the general formula that covers all of them at once, and which in hindsight makes the individual cases corollaries rather than separate computations.

1.2 Setting and notation

Throughout, the solid is a *regular n -prism*: two congruent regular n -gons joined by n rectangular lateral faces.

Definition 1.1 (Parameters). For a regular n -gon of side s we write

$$r = \frac{s}{2} \cot \frac{\pi}{n} \quad (\text{apothem, the inradius}), \quad R = \frac{s}{2 \sin(\pi/n)} \quad (\text{circumradius}),$$

so that $r = R \cos(\pi/n)$. The *base height* b is the length of the bisector of the base, that is, the segment joining two opposite boundary points through the centre:

$$b = \begin{cases} r + R = \frac{s}{2} \cot \frac{\pi}{2n}, & n \text{ odd} \quad (\text{vertex to midpoint of the opposite side}), \\ 2r = s \cot \frac{\pi}{n}, & n \text{ even} \quad (\text{midpoint to midpoint of opposite sides}). \end{cases}$$

The height of the prism is written $h = kb$, so that k is the height measured in units of the base height. Finally $\beta = 2\pi/n$.

The distinction between odd and even n in Definition 1.1 is not a bookkeeping nuisance. It turns out to be the single source of every difference between odd and even prisms in this problem, and it is worth giving it a name.

Definition 1.2 (Asymmetry parameter). Put $c := b - r$ and

$$\varepsilon := c - r = b - 2r = \begin{cases} R - r = \frac{s}{2} \tan \frac{\pi}{2n} > 0, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

For odd n the two ends of the bisector are of different kinds — one is a vertex, at distance R from the centre, the other the midpoint of a side, at distance r — and ε measures that difference. For even n both ends are midpoints of sides and ε vanishes. Theorem 8.6 and Theorem 8.7 will show that everything hinges on this.

1.3 The two points and the obvious path

Definition 1.3 (The ant and the honey). Let $p \in [0, \frac{1}{2}]$. The ant sits at the point A of the lower base which lies on the bisector of that base, at distance $b(1 - p)$ from one end of the bisector; equivalently, at distance bp from the other end. The honey sits at the point B of the upper base, on the bisector of that base, at distance bp from the end of the bisector that lies vertically above the end of the ant's bisector nearest the honey.

In words: both points lie on the bisector, both at the same relative position, one measured from the near rim and the other from the far rim, so that the configuration is symmetric under turning the prism upside down. The value $p = 0$ puts the ant on the rim; $p = \frac{1}{2}$ puts it at the midpoint of the bisector.

Remark 1.4 (The obvious path). The *obvious path* runs from A along the bisector of the lower base to the rim (length $b(1 - p)$), then vertically up a lateral face (length h), then along the bisector of the upper base to B (length bp). Its total length is

$$D = b(1 - p) + h + bp = b + h = b(k + 1),$$

which does not depend on p . Every question below is whether some path beats D .

1.4 Method: unfolding and rolling

All computations use the same device. Cut the lateral surface along one vertical edge and lay it flat: it becomes a horizontal strip

$$\Pi = \{(x, y) : Y_b \leq y \leq Y_t\}, \quad Y_b = b, \quad Y_t = b + h,$$

which may be extended indefinitely to the left and right, one copy of the strip per circuit of the prism. The two bases are then attached to the strip along one side each. Since a path may leave the lower base through any of its n sides and enter the upper base through any of its n sides, we represent all possibilities by *rolling* the bases along the strip: D_i denotes the lower base after i rolling steps to the right, G_j the upper base after j steps. A path from A to B that crosses the lower base, the lateral surface and the upper base becomes, in this picture, a straight segment from A_i to B_j — provided it stays inside the region actually occupied by material.

That proviso is the whole difficulty, and Section 8 states it exactly. Two facts about rolling will be used repeatedly and are recorded here.

Lemma 1.5 (Rolling). *After i steps the contact side of D_i occupies the interval $[is, (i + 1)s]$ on the line $y = Y_b$; the centre of D_i is at $((i + \frac{1}{2})s, Y_b - r)$; and the map carrying D_0 to D_i is a direct isometry whose rotation angle is $i\beta$. The corresponding statements hold for G_j , with contact side $[js, (j + 1)s]$ on $y = Y_t$, centre $((j + \frac{1}{2})s, Y_t + r)$, and rotation angle $-j\beta$.*

Proof. Induction on i . Each step is a rotation through the exterior angle β about the right-hand endpoint of the current contact side; it brings the next side down onto the adjacent interval of length s , and the convex polygon stays in the half-plane $y \leq Y_b$. The centre of a regular polygon is at distance r from each side, and its foot of perpendicular on a side is that side's midpoint, which gives the stated position. A composition of i rotations through β is a direct isometry with rotation angle $i\beta$. $\square$

Throughout, a *class* of shorter paths means a set of paths of equal length. Paths come in mirror pairs — the ant may set off to the left or to the right — and the two members of a pair always have the same length, so a class normally contains two paths. When we say a prism “has four shorter paths” we mean four classes.

2 The square prism: the original puzzle

Take $n = 4$ with side $s = 4$ and height $h = 10$. For $n = 4$ the base height equals the side, $b = s \cot(\pi/4) = s = 4$, so the obvious path has length

$$D = h + b = 10 + 4 = 14.$$

The ant is 1 unit from the top edge of its face, the honey 1 unit from the bottom edge of the opposite face, both centred horizontally — that is, $p = \frac{1}{4}$ in the notation of Definition 1.3. See Figure 1.

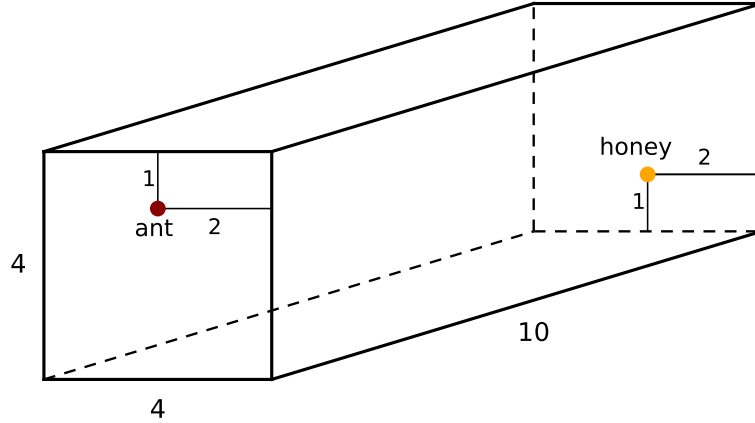


Figure 1: The original puzzle: a $4 \times 4 \times 10$ box. The obvious route has length $4 + 10 = 14$.

Unfold. Lay the four lateral faces flat as a strip of width $4s = 16$ and height 10, and attach the two bases. Now roll: turn the upper square through 90° about the rim, so that it comes to rest one panel further along the strip. The honey travels with it. The straight segment from the ant to the rolled honey has horizontal extent $s/2 + s/2 + s = \dots$; concretely, the two points are separated by 13 vertically and 5 horizontally, so

$$d = \sqrt{(10 + 3)^2 + 5^2} = \sqrt{194} = 13.9284\dots < 14 = \sqrt{196}.$$

So the obvious path is not optimal: the ant saves about 0.07 by setting off diagonally.

Two remarks are worth making at once, because both grow into general phenomena.

First, the shorter path is not unique. It is immaterial whether the ant turns left or right, and immaterial whether the upper square is rolled clockwise or counter-clockwise; all four combinations give segments of the same length $\sqrt{194}$. In the language of Section 1 this is one *class* containing four paths — two mirror pairs.

Second, the computation above is not yet a proof that $\sqrt{194}$ is the minimum. It shows only that one particular unfolding beats the obvious path. A complete argument must enumerate the unfoldings and, crucially, check that each candidate segment really lies on the surface of the prism. That check is where most of the subtlety of the general problem turns out to live; it is stated precisely in Section 8.

For this concrete case the answer is that $\sqrt{194}$ is indeed the minimum, and it is not hard to see that a shorter path cannot pass through a vertex of the prism.

3 The square prism in general

We now let s , h and p vary. Since $b = s$ for $n = 4$, the obvious path has length

$$D = s + h = s(k + 1), \quad k = h/s.$$

3.1 When does a shorter path exist at all?

Start with the ant on the rim, $p = 0$. Write d_0 for the length of the best slanted path in that position. It is obtained by rolling one square through 180° rather than 90° , which puts the two points $2s$ apart horizontally and h apart vertically:

$$d_0^2 = h^2 + (2s)^2 = h^2 + 4s^2.$$

This beats the obvious path exactly when $d_0^2 < D^2 = h^2 + 2sh + s^2$, that is when $4s^2 < 2sh + s^2$, i.e.

$$h > \frac{3}{2}s, \quad \text{equivalently} \quad k > \frac{3}{2}. \quad (1)$$

So the square prism must be at least one and a half times as tall as it is wide before any saving is possible at all. This is a genuine restriction, and it is peculiar to small prisms: Section 9 shows that the corresponding threshold decreases with n and that for odd n it disappears entirely.

3.2 Two competing paths

For $0 < p \leq \frac{1}{2}$ there are two candidate classes, obtained by rolling through 90° and through 180° respectively. In the index notation of Section 8 they are the pairs $(i, j) = (2, 1)$ and $(2, 0)$. Their lengths are

$$d_1 = s\sqrt{k^2 + k + 2kp + 2p^2 - 2p + \frac{5}{2}}, \quad (2)$$

$$d_2 = s\sqrt{k^2 + 4kp + 4p^2 + 4}. \quad (3)$$

Formula (2) comes from the horizontal and vertical components $(\frac{s}{2} + s(1-p))$ and $(sp + sk + \frac{s}{2})$; formula (3) from $(\frac{s}{2} + s + \frac{s}{2})$ and $(sp + sk + sp)$.

Which of the two is shorter depends on p . Setting $d_1 = d_2$ gives

$$2p^2 + 2(k+1)p + \frac{3}{2} - k = 0, \quad p_{eq} = \frac{1}{2} \left(\sqrt{k^2 + 4k - 2} - k - 1 \right), \quad (4)$$

or, solved the other way, $k_{eq} = \frac{p^2 + p + 3/4}{1/2 - p}$. For $p \leq p_{eq}$ the path d_2 is the shorter, for $p \geq p_{eq}$ the path d_1 . With $k = 3.5$ one finds $p_{eq} = 0.2122\dots$

Remark 3.1. For $2 \leq k \leq 5$ these two are the only classes. For larger k further classes appear; at $k = 10$, for instance, there is a third. Section 8 explains where they come from and Appendix C shows that their number grows without bound as $k \rightarrow \infty$, provided helical paths are admitted.

3.3 How far from the rim can the ant be?

Each class beats the obvious path only for p below a threshold. Imposing $d_1 \leq D$ in (2) gives $2p^2 + 2(k-1)p + \frac{3}{2} - k \leq 0$ and hence

$$p_{max1} = \frac{1}{2} \left(\sqrt{k^2 - 2} - k + 1 \right), \quad (5)$$

while $d_2 \leq D$ gives $2p^2 + 2kp + \frac{3}{2} - k \leq 0$ and

$$p_{max2} = \frac{1}{2} \left(\sqrt{k^2 + 2k - 3} - k \right). \quad (6)$$

For $k = 3.5$: $p_{max1} = 0.3507\dots$ and $p_{max2} = 0.2655\dots$ Note that beyond p_{eq} the class d_2 is no longer the shortest, but it remains shorter than the obvious path up to p_{max2} .

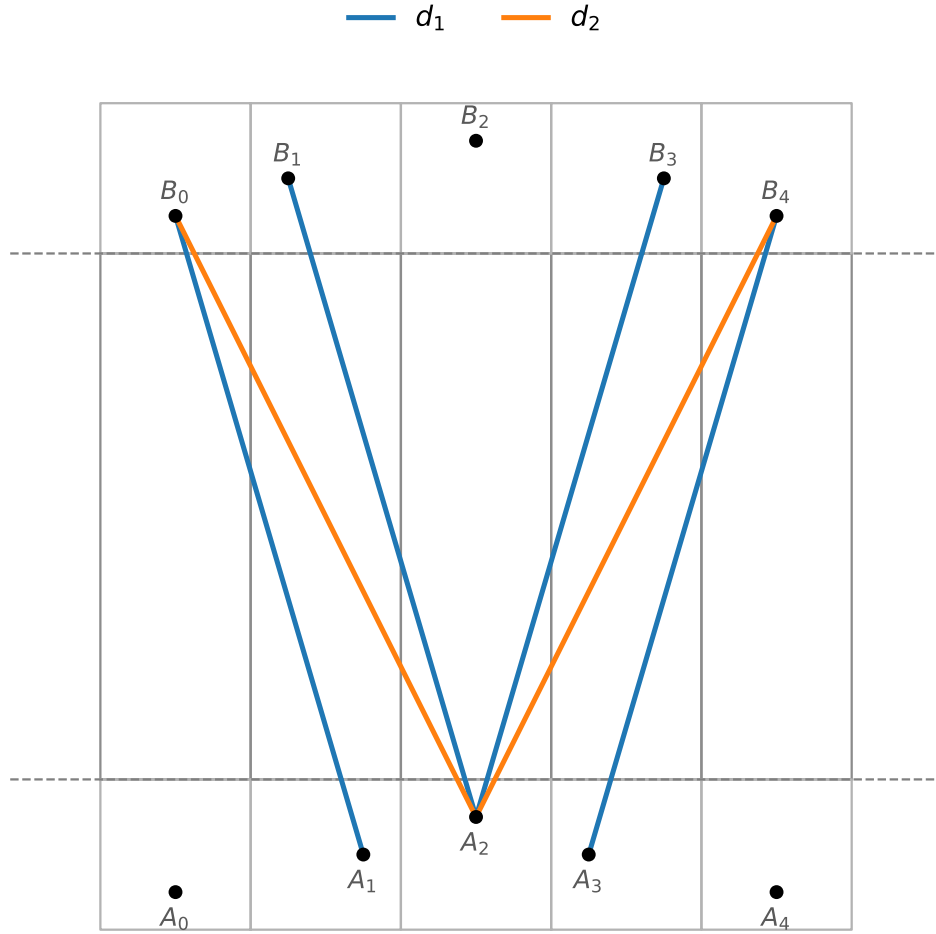


Figure 2: The square prism unfolded, $b = s = 4$, $k = 3.5$, $p = 0.25$: the lateral surface as a strip, the two bases rolled into all their positions, and the two classes of shorter paths in colour.

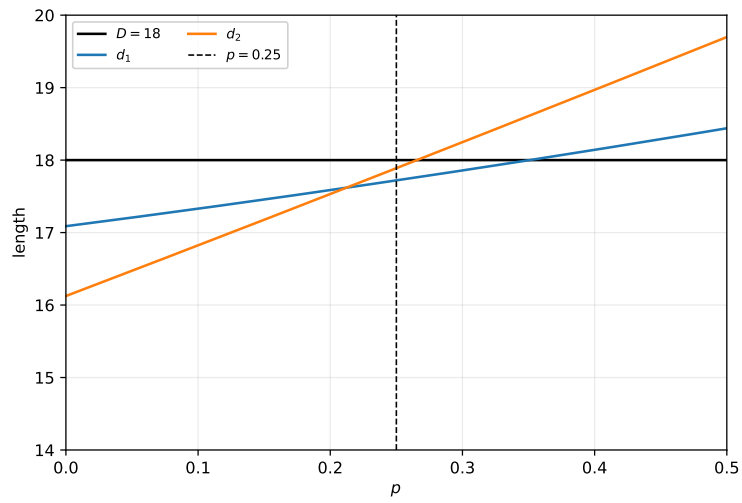


Figure 3: The two lengths on the square prism as functions of p , against the obvious path $D = 18$ ($b = 4$, $k = 3.5$).

3.4 The most efficient shape

It is natural to ask which k gives the largest relative saving, that is, minimises d/D . For the branch d_1 ,

$$\frac{d_1}{D} = \frac{\sqrt{k^2 + k + 2kp + 2p^2 - 2p + 5/2}}{k + 1},$$

and differentiating with respect to k and clearing the numerator gives $(2p-1)k + 4p^2 - 6p + 4 = 0$, hence

$$k_{opt1} = \frac{2p^2 - 3p + 2}{1/2 - p}, \quad \text{and similarly} \quad k_{opt2} = \frac{2p^2 - p + 2}{1/2 - p}. \quad (7)$$

Equating k_{eq} with k_{opt1} gives the changeover

$$p_{opt} = 2 - \frac{\sqrt{11}}{2} = 0.3416\dots,$$

below which k_{opt2} applies and above which k_{opt1} does. (Equating k_{eq} with k_{opt2} has no real solution.)

3.5 At what angle should the ant set off?

Measuring from the horizontal in the ant's own base — the convention is made precise in Section 8 — the two classes require

$$\theta_1 = \arctan \frac{2k + 2p + 1}{3 - 2p}, \quad \theta_2 = \arctan \frac{k + 2p}{2}. \quad (8)$$

With $k = 3.5$, $p = 0.25$ these give 73.61° and 63.43° .

3.6 Summary for $2 \leq k \leq 5$

$D = s(k + 1)$	$k > 3/2$ required for any shorter path
$d_1 = s\sqrt{k^2 + k + 2kp + 2p^2 - 2p + 5/2}$	valid for $p \geq p_{eq}$
$d_2 = s\sqrt{k^2 + 4kp + 4p^2 + 4}$	valid for $p \leq p_{eq}$
$p_{eq} = \frac{1}{2}(\sqrt{k^2 + 4k - 2} - k - 1)$	changeover
$p_{max1} = \frac{1}{2}(\sqrt{k^2 - 2} - k + 1)$	
$p_{max2} = \frac{1}{2}(\sqrt{k^2 + 2k - 3} - k)$	
$k_{opt1} = (2p^2 - 3p + 2)/(1/2 - p)$	for $p \geq p_{opt}$
$k_{opt2} = (2p^2 - p + 2)/(1/2 - p)$	for $p \leq p_{opt}$
$p_{opt} = 2 - \sqrt{11}/2$	
$\theta_1 = \arctan \frac{2k+2p+1}{3-2p}, \quad \theta_2 = \arctan \frac{k+2p}{2}$	

Remark 3.2. One feature of the square prism is worth flagging now, because it is exceptional. Both classes exist, as paths on the surface, for *every* $p \in [0, \frac{1}{2}]$. For most other prisms this fails: certain paths simply do not lie on the surface when the ant is too near the rim, and only come into existence above a threshold value of p . See Section 5 for the first example and Section 9 for the general picture. Because the square prism was the starting point, this phenomenon is easy to overlook.

4 The triangular prism

For $n = 3$ the base is an equilateral triangle and its bisector runs from a vertex to the midpoint of the opposite side, so

$$b = v = \frac{s\sqrt{3}}{2}, \quad D = h + b = b(k + 1).$$

(For odd n in general, $b = \frac{s}{2} \cot \frac{\pi}{2n}$.)

4.1 Four classes

At $k = 3.5$ and $p = 0.25$ the triangular prism admits four classes of shorter paths, with index pairs $(1, 1)$, $(1, 2)$, $(1, 0)$ and $(1, 3)$. Their lengths are

$$d_1 = b\sqrt{k^2 + k + 3p^2 - 3p + 1}, \quad (9)$$

$$d_2 = b\sqrt{k^2 + k + 1}, \quad (10)$$

$$d_3 = b\sqrt{k^2 + 3kp + 3p^2 - 3p + 3}, \quad (11)$$

$$d_4 = b\sqrt{k^2 + 3kp + 3p^2 + 3p + 3}. \quad (12)$$

Two features stand out.

Remark 4.1 (A path that does not care where the ant stands). The length d_2 contains no p at all. Whether the ant sits on the rim or at the middle of the bisector, this path has the same length $b\sqrt{k^2 + k + 1}$. It is the first instance of what Section 10 calls a *p-independent* path, and Appendix A shows that every odd prism has exactly one such path shorter than D , of length $b\sqrt{k^2 + 2k \cos(\pi/n) + 1}$; for $n = 3$, $2 \cos(\pi/3) = 1$.

Remark 4.2 (No height threshold). Both d_1 and d_2 are shorter than D for *every* $k > 0$ and every p . Indeed

$$d_2^2 - D^2 = b^2(k^2 + k + 1 - (k + 1)^2) = -b^2k < 0.$$

The triangular prism therefore never needs to be tall before a saving appears — in sharp contrast with the square prism, which requires $k > 3/2$. Section 9 explains the reason.

The other two classes do carry a threshold in k :

$$d_3 < D \iff k > \frac{3p^2 - 3p + 2}{2 - 3p}, \quad d_4 < D \iff k > \frac{3p^2 + 3p + 2}{2 - 3p},$$

with

$$p_{max3} = \frac{1}{6} \left(\sqrt{9k^2 + 6k - 15} - 3k + 3 \right), \quad p_{max4} = \frac{1}{6} \left(\sqrt{9k^2 + 42k - 15} - 3k - 3 \right).$$

4.2 Which is shortest

The ordering of $d_1, \dots, d_4$ changes with p and k . The changeovers are

$$\begin{aligned} p_{eq,1-3} &= \frac{1}{3} \left(1 - \frac{2}{k} \right) \quad (\text{requires } k > 2), & \text{equivalently } k_{eq,1-3} &= \frac{2}{1 - 3p}, \\ p_{eq,1-4} &= \frac{k - 2}{3k + 6}, & p_{eq,2-3} &= \frac{1}{6} \left(\sqrt{9k^2 - 6k - 15} - 3k + 3 \right), \\ p_{eq,2-4} &= \frac{1}{6} \left(\sqrt{9k^2 + 30k - 15} - 3k - 3 \right). \end{aligned}$$

If $k > 2$ and $p < \frac{1}{3}(1 - 2/k)$ then d_3 is shorter than d_1 ; otherwise d_1 is shorter. Comparing d_1 with d_2 : they agree at $p = 0$ and $d_1 < d_2$ for all $p > 0$. Comparing d_3 with d_4 : they agree at $p = 0$ and $d_3 < d_4$ for all $p > 0$.

4.3 A fifth class

The list above is complete only for moderate p . A fifth class, with index pair $(0, 1)$, has length

$$d_5 = b\sqrt{k^2 + 3k(1 - p) + 3p^2 - 3p + 3}, \quad (13)$$

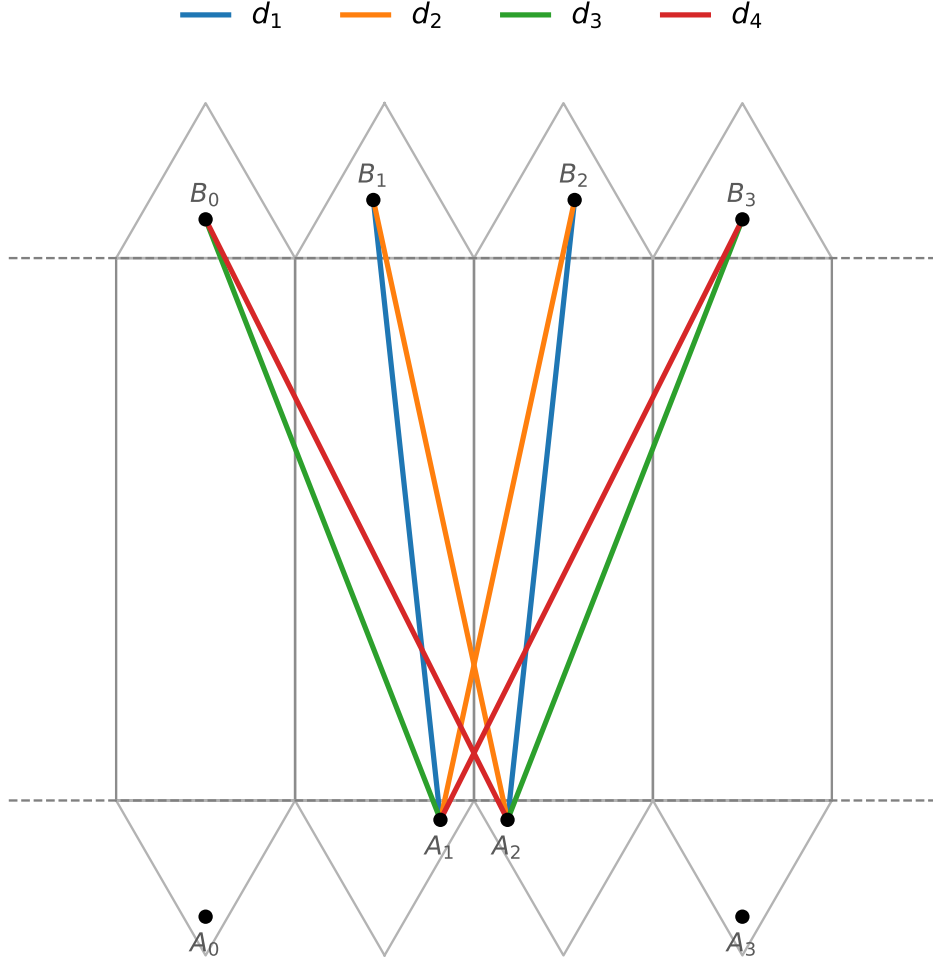


Figure 4: The triangular prism unfolded, $b = 4$, $k = 3.5$, $p = 0.25$: the four classes of shorter paths.

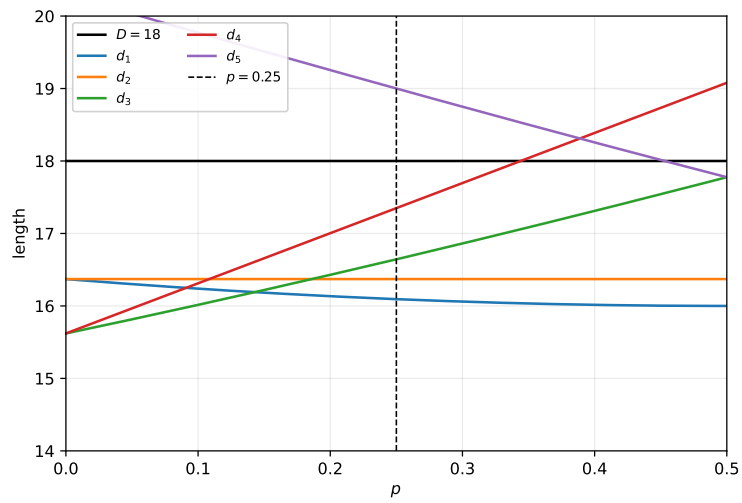


Figure 5: Lengths on the triangular prism against p ($b = 4$, $k = 3.5$). The curve d_2 is horizontal because it does not depend on p ; the fifth class d_5 of Section 4.3 descends and crosses D at $p = 0.453$.

and comparison with (11) shows the pleasant relation

$$d_5(p) = d_3(1 - p).$$

This path always lies on the surface, but it is shorter than the obvious path only for

$$p \geq p_{min5} = \frac{k+1}{2} - \frac{\sqrt{9k^2 + 6k - 15}}{6} = 1 - p_{max3},$$

which for $k = 3.5$ gives $p_{min5} = 0.45301 \dots$. So for $k = 3.5$ the triangular prism has four classes of shorter paths when $p < 0.453$ and, in the narrow band above that, a fifth. (At $p = \frac{1}{2}$ the classes (0, 1) and (1, 0) coincide in length, as they must by symmetry.)

4.4 Angles

With the convention of Section 8,

$$\begin{aligned} \theta_1 &= \arctan\left(\frac{\sqrt{3}}{3} \cdot \frac{2k+1}{1-2p}\right) - \frac{\pi}{3}, & \theta_2 &= \frac{2\pi}{3} - \arctan\left(\frac{\sqrt{3}}{3}(2k+1)\right), \\ \theta_3 &= \arctan\left(\frac{\sqrt{3}}{3} \cdot \frac{2k+3p}{2-p}\right) - \frac{\pi}{3}, & \theta_4 &= \frac{2\pi}{3} - \arctan\left(\frac{\sqrt{3}}{3} \cdot \frac{6k+9p}{6+3p}\right). \end{aligned}$$

At $k = 3.5$, $p = 0.25$ these are 23.82° , 42.22° , 8.64° and 56.70° respectively.

4.5 Summary for $2 \leq k \leq 5$

$D = b(k+1), \quad b = s\sqrt{3}/2$	no height threshold: $d_1, d_2 < D$ for all $k > 0$
$d_1 = b\sqrt{k^2 + k + 3p^2 - 3p + 1}$	
$d_2 = b\sqrt{k^2 + k + 1}$	independent of p
$d_3 = b\sqrt{k^2 + 3kp + 3p^2 - 3p + 3}$	$p \leq p_{max3}$
$d_4 = b\sqrt{k^2 + 3kp + 3p^2 + 3p + 3}$	$p \leq p_{max4}$
$d_5 = b\sqrt{k^2 + 3k(1-p) + 3p^2 - 3p + 3}$	$p \geq 1 - p_{max3}$
$p_{max3} = \frac{1}{6}(\sqrt{9k^2 + 6k - 15} - 3k + 3)$	
$p_{max4} = \frac{1}{6}(\sqrt{9k^2 + 42k - 15} - 3k - 3)$	

5 The hexagonal prism

For $n = 6$ the bisector joins the midpoints of two opposite sides, so with $v = s\sqrt{3}/2$ the height of one of the six equilateral triangles,

$$b = 2v = s\sqrt{3} = s \cot \frac{\pi}{6}, \quad D = h + b = b(k+1).$$

(For even n in general, $b = s \cot(\pi/n)$.)

5.1 Four classes

At $k = 3.5$, $p = 0.25$ there are four classes, with index pairs (3, 1), (3, 0), (2, 1) and (3, 2):

$$d_1 = b\sqrt{k^2 + \frac{1}{2}k + 3kp + 3p^2 - 2p + \frac{31}{12}}, \quad (14)$$

$$d_2 = b\sqrt{k^2 + 4kp + 4p^2 + 3}, \quad (15)$$

$$d_3 = b\sqrt{k^2 + k + 2kp + 4p^2 - 4p + \frac{7}{3}}, \quad (16)$$

$$d_4 = b\sqrt{k^2 + \frac{3}{2}k + kp + p^2 - p + \frac{19}{12}}. \quad (17)$$

Each is shorter than the obvious path only above a threshold in k :

$$\begin{aligned} d_1 < D &\iff k > \frac{2p^2 - \frac{4}{3}p + \frac{19}{18}}{1 - 2p}, & d_2 < D &\iff k > \frac{2p^2 + 1}{1 - 2p}, \\ d_3 < D &\iff k > \frac{4}{3} \cdot \frac{3p^2 - 3p + 1}{1 - 2p}, & d_4 < D &\iff k > \frac{2p^2 - 2p + \frac{7}{6}}{1 - 2p}, \end{aligned}$$

with the corresponding upper limits in p

$$\begin{aligned} p_{max1} &= \frac{1}{6}(\sqrt{9k^2 + 6k - 15} - 3k + 2), & p_{max2} &= \frac{1}{2}(\sqrt{k^2 + 2k - 2} - k), \\ p_{max3} &= \frac{1}{12}(\sqrt{9k^2 - 12} - 3k + 6), & p_{max4} &= \frac{1}{6}(\sqrt{9k^2 - 12} - 3k + 3). \end{aligned}$$

The changeover values are

$$\begin{aligned} p_{eq,1-2} &= \frac{1}{6}(\sqrt{9k^2 + 54k + 21} - 3k - 6), & p_{eq,1-3} &= \frac{1}{2}(-\sqrt{k^2 + 2k + 5} + k + 2), \\ p_{eq,1-4} &= \frac{1}{4}(\sqrt{4k^2 + 4k - 7} - 2k + 1), & p_{eq,2-3} &= \frac{3k - 2}{6k + 12}, \\ p_{eq,2-4} &= \frac{1}{6}(\sqrt{9k^2 + 24k - 16} - 3k - 1), & p_{eq,3-4} &= \frac{1}{2} - \frac{k}{3} \quad (k < \frac{3}{2}), \text{ else } \frac{1}{2}. \end{aligned}$$

Remark 5.1. Unlike the triangular prism, the hexagonal one has no p -independent path shorter than D . This is not an accident of the formulas: Theorem 8.6 shows that no even prism has one. The reason is the asymmetry parameter ε of Definition 1.2, which vanishes precisely when n is even.

5.2 Paths that do not exist for small p

Here the hexagonal prism shows something the square and triangular ones do not. Two of the four classes are not merely longer than D for small p — they *do not exist at all*. The straight segment in the unfolded picture leaves the base through the wrong side, so it does not correspond to any path on the surface of the prism.

Writing the exact feasibility criterion of Section 8 as a condition on p produces, for each class, a quadratic inequality. Solving it gives:

$$d_3 = (2, 1) : \quad p \geq p_{min3} = \frac{k - 2}{2(3k + 2)}, \quad \text{a threshold only for } k > 2, \quad (18)$$

$$d_1 = (3, 1) : \quad p \geq p_{min1} = \frac{\sqrt{9k^2 + 18k - 11}}{6} - \frac{k}{2} - \frac{1}{3}, \quad \text{only for } k > \frac{5}{2}. \quad (19)$$

The classes d_2 and d_4 exist for every p .

For $k = 3.5$ these evaluate to

$$p_{min1} = \frac{\sqrt{649} - 25}{12} = 0.0396232\dots, \quad p_{min3} = \frac{3}{50} = 0.06 \text{ exactly,}$$

the underlying quadratics being $6p^2 + 25p - 1 = 0$ and $50p - 3 = 0$ respectively.

Remark 5.2 (Why one is linear and the other is not). The leading coefficient of both feasibility quadratics equals $-\sin((i + j)\beta)$ (Proposition 8.5). For d_3 we have $i + j = 3$ and $6 \mid 2 \cdot 3$, so the coefficient vanishes and the condition is linear, giving a rational threshold. For d_1 , $i + j = 4$ and the condition is genuinely quadratic, so a square root appears.

Remark 5.3 (What the threshold means). Geometrically the ant, when close to the rim, is far from the centre of its base; the straight line towards the honey then exits the base through a neighbouring side rather than the intended one. As p grows the ant moves towards the centre, the line steepens relative to the base, and at $p = p_{min}$ the exit point enters the correct side. The same picture explains why the thresholds increase with k : a taller prism means a line that is more slanted relative to the base.

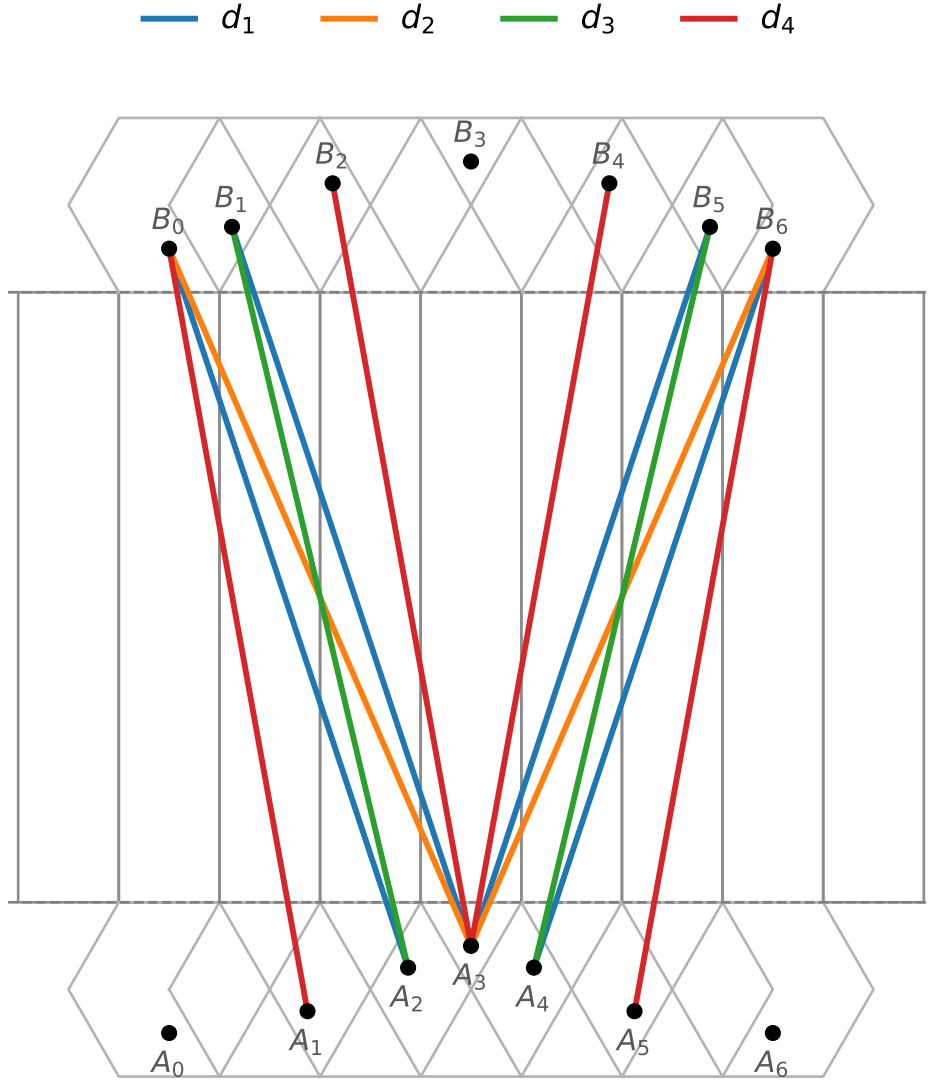


Figure 6: The hexagonal prism unfolded, $b = 4$, $k = 3.5$, $p = 0.25$: the four classes of shorter paths.

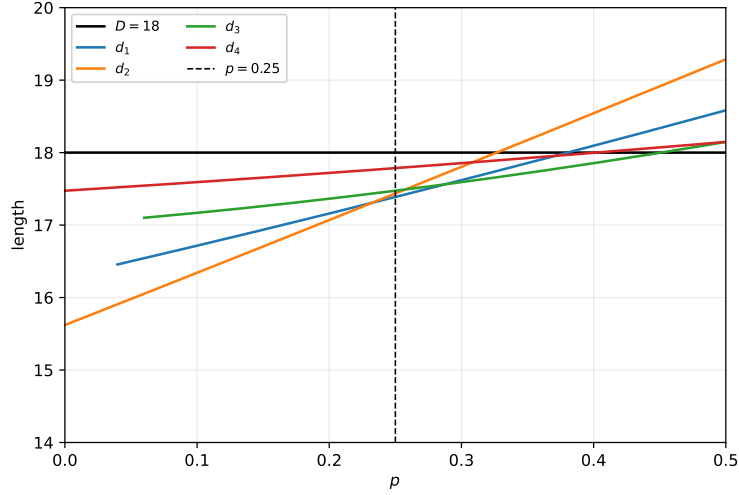


Figure 7: Lengths on the hexagonal prism against p ($b = 4$, $k = 3.5$). The curves d_1 and d_3 begin only at $p = 0.0396$ and $p = 0.06$; below those values the corresponding paths do not lie on the surface at all (Section 5.2).

5.3 Angles

$$\begin{aligned}\theta_1 &= \arctan \frac{\sqrt{3}(4k + 6p + 1)}{11 - 6p}, & \theta_2 &= \arctan \left(\frac{\sqrt{3}}{3}(k + 2p) \right), \\ \theta_3 &= \arctan \frac{\sqrt{3}(2k + 2p + 1)}{5 - 6p} - \frac{\pi}{3}, & \theta_4 &= \arctan \frac{\sqrt{3}(4k + 2p + 3)}{7 - 6p}.\end{aligned}$$

At $k = 3.5$, $p = 0.25$: 71.61° , 66.59° , 16.63° , 79.71° .

6 The pentagonal prism

For $n = 5$ the bisector joins a vertex to the midpoint of the opposite side, and

$$b = \frac{s}{2} \cot \frac{\pi}{10} = v + t, \quad D = h + b = b(k + 1),$$

where v and t are the height and the leg of one of the five triangles making up the pentagon.

The pentagonal prism is the richest of the small cases: at $k = 3.5$, $p = 0.25$ it admits *eight* classes of shorter paths, twice as many as the triangular or hexagonal prism. All eight are given below.

6.1 The eight classes

Ordered by length at $k = 3.5$, $p = 0.25$, the classes have the index pairs listed in Table 1. Each has a mirror twin $(5 - i, 5 - j)$ of the same length.

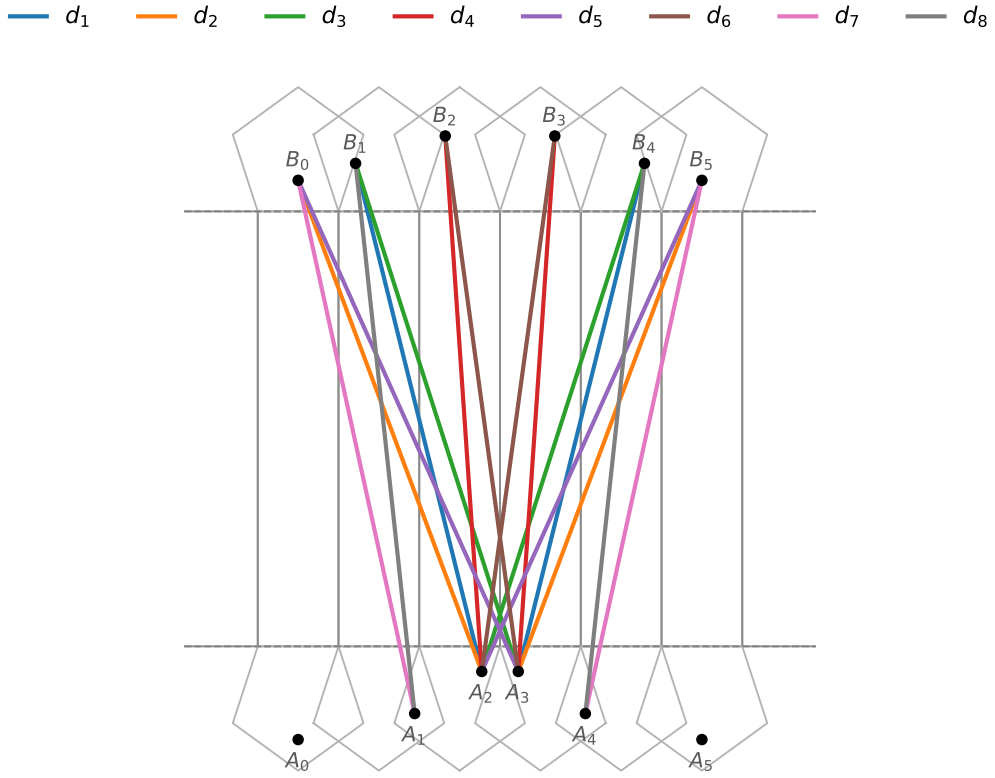


Figure 8: The pentagonal prism at $b = 4$, $k = 3.5$, $p = 0.25$: all eight classes in the unfolded picture, with the bases rolled into every position.

class	(i, j)	mirror	d at $b = 4, k = 3.5, p = 0.25$	θ
d_1	(2, 1)	(3, 4)	16.850887	40.053°
d_2	(2, 0)	(3, 5)	16.877805	33.500°
d_3	(2, 4)	(3, 1)	17.172173	71.757°
d_4	(2, 2)	(3, 3)	17.276111	50.098°
d_5	(2, 5)	(3, 0)	17.324519	78.144°
d_6	(2, 3)	(3, 2)	17.395686	61.768°
d_7	(1, 0)	(4, 5)	17.568277	-30.329°
d_8	(1, 1)	(4, 4)	17.810068	-24.131°

Table 1: The eight classes on the pentagonal prism.

Writing $d_t = b\sqrt{Q_t}$, the eight length formulas are

$$Q_1 = k^2 + \frac{\sqrt{5}-1}{2}k + \sqrt{5}kp + \frac{5+\sqrt{5}}{2}p^2 - \frac{5+\sqrt{5}}{2}p + (11 - 4\sqrt{5}), \quad (20)$$

$$Q_2 = k^2 + \frac{5+\sqrt{5}}{2}kp + \frac{5+\sqrt{5}}{2}p^2 + \frac{5\sqrt{5}-15}{2}p + (25 - 10\sqrt{5}), \quad (21)$$

$$Q_3 = k^2 + \frac{\sqrt{5}-1}{2}k + \sqrt{5}kp + \frac{5-\sqrt{5}}{2}p^2 + \frac{15-7\sqrt{5}}{2}p + (11 - 4\sqrt{5}), \quad (22)$$

$$Q_4 = k^2 + \frac{1+\sqrt{5}}{2}k + \frac{5-\sqrt{5}}{2}p^2 - \frac{5-\sqrt{5}}{2}p + 1, \quad (23)$$

$$Q_5 = k^2 + \frac{5+\sqrt{5}}{2}kp + \frac{5+\sqrt{5}}{2}p^2 + \frac{15-5\sqrt{5}}{2}p + (25 - 10\sqrt{5}), \quad (24)$$

$$Q_6 = k^2 + \frac{1+\sqrt{5}}{2}k + 1, \quad (25)$$

$$Q_7 = k^2 + (\sqrt{5} - 1)k + \frac{5-\sqrt{5}}{2}kp + \frac{5-\sqrt{5}}{2}p^2 - \frac{5-\sqrt{5}}{2}p + (4 - \sqrt{5}), \quad (26)$$

$$Q_8 = k^2 + \frac{3(\sqrt{5}-1)}{2}k + \frac{5+\sqrt{5}}{2}p^2 - \frac{5+\sqrt{5}}{2}p + (4 - \sqrt{5}). \quad (27)$$

Note (25): the class d_6 carries no p . It is the p -independent path of Appendix A, and indeed $\frac{1+\sqrt{5}}{2} = 2\cos(\pi/5)$, so $d_6 = b\sqrt{k^2 + 2k\cos(\pi/5) + 1}$.

6.2 Angles

Write $\Delta x/b = X_0 + X_1p$ and $\Delta y/b = k + Y_0 + Y_1p$ for the components of the segment in the unfolded picture. Then, with $\beta = 2\pi/5$,

$$\theta_t = \begin{cases} \arctan \frac{\Delta y}{|\Delta x|} - \frac{\pi}{5}, & t \in \{1, 2, 4\}, \\ \frac{4\pi}{5} - \arctan \frac{\Delta y}{\Delta x}, & t \in \{3, 5, 6\}, \\ \arctan \frac{\Delta y}{|\Delta x|} - \frac{3\pi}{5}, & t \in \{7, 8\}, \end{cases}$$

the three cases corresponding to the sign of Δx and the value of i ; see Section 8. On $p \in [0, \frac{1}{2}]$ the sign of Δx never changes within a class, so each class uses one and the same case throughout. The coefficients are given in Table 2.

Remark 6.1 (Negative angles). The classes d_7 and d_8 have $\theta < 0$ at $k = 3.5, p = 0.25$. This is not an artefact of choosing one mirror twin rather than the other: the twins give θ and $180^\circ - \theta$, and both stay negative. A negative θ simply means that the ant sets off on the other side of the reference direction of its base compared with classes $d_1, \dots, d_6$.

class	X_0	X_1	Y_0	Y_1
d_1	$-\frac{10 - 3\sqrt{5}}{\sqrt{10 - 2\sqrt{5}}}$	$\frac{\sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{5}}{2}$
d_2	$-\sqrt{25 - 10\sqrt{5}}$	$\frac{\sqrt{10 - 2\sqrt{5}}}{4}$	0	$\frac{5+\sqrt{5}}{4}$
d_3	$+\frac{10 - 3\sqrt{5}}{\sqrt{10 - 2\sqrt{5}}}$	$\frac{\sqrt{10 - 2\sqrt{5}} - \sqrt{10 + 2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{5}}{2}$
d_4	$-\frac{\sqrt{10 - 2\sqrt{5}}}{4}$	$\frac{\sqrt{10 - 2\sqrt{5}}}{2}$	$\frac{1+\sqrt{5}}{4}$	0
d_5	$+\sqrt{25 - 10\sqrt{5}}$	$\frac{\sqrt{10 - 2\sqrt{5}}}{4}$	0	$\frac{5+\sqrt{5}}{4}$
d_6	$+\frac{\sqrt{10 - 2\sqrt{5}}}{4}$	0	$\frac{1+\sqrt{5}}{4}$	0
d_7	$-\frac{\sqrt{10 - 2\sqrt{5}}}{2}$	$\frac{\sqrt{10 + 2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{2}$	$\frac{5-\sqrt{5}}{4}$
d_8	$-\frac{\sqrt{10 + 2\sqrt{5}}}{4}$	$\frac{\sqrt{10 + 2\sqrt{5}}}{2}$	$\frac{3(\sqrt{5}-1)}{4}$	0

Table 2: Components of the eight classes, $\Delta x/b = X_0 + X_1 p$, $\Delta y/b = k + Y_0 + Y_1 p$.

6.3 Thresholds and ranges

Two of the eight classes do not exist for small p , in the sense of Section 5.2:

$$d_7 : p \geq 0.035106 \dots \text{ (a threshold only for } k > 3), \quad d_8 : p \geq 0.107422 \dots \text{ (only for } k > 2).$$

The binding condition is in both cases a quadratic $Ap^2 + Bp + C = 0$ with

$$d_7 : A = \sin \frac{2\pi}{5}, \quad B = 0.9510565 k - 0.5020285, \quad C = 0.6024343 - 0.2008114 k,$$

$$d_8 : A = \sin \frac{\pi}{5}, \quad B = 0.9510565 k - 0.5877853, \quad C = 0.4016228 - 0.2008114 k,$$

and the condition $C < 0$ is exactly $k > 3$, respectively $k > 2$.

Combining feasibility with the requirement $d < D$, the intervals of p on which each class is an actual shorter path are, at $k = 3.5$:

class	from	to	class	from	to
d_1	0	0.5	d_5	0	0.33927
d_2	0	0.43488	d_6	0	0.5
d_3	0	0.46441	d_7	0.03511	0.46594
d_4	0	0.5	d_8	0.10742	0.5

6.4 The graph

Figure 9 plots all eight lengths against p for $b = 4$, $k = 3.5$, together with the obvious path $D = 18$. The curves for d_7 and d_8 begin only at their thresholds.

Remark 6.2 (The numbering is local). The labels $d_1, \dots, d_8$ order the classes by length at $k = 3.5$, $p = 0.25$, and that ordering is not preserved elsewhere. The figure shows it directly: near $p = 0$ the curves d_5 and d_2 lie *below* d_1 . The same happens as k varies; at $k = 2$ and small p the order is $d_2, d_5, d_1, d_3, d_4, d_6, d_7, d_8$. For anything beyond a fixed (k, p) it is therefore safer to name a class by its index pair (i, j) , which is structural, and to quote the rank separately.

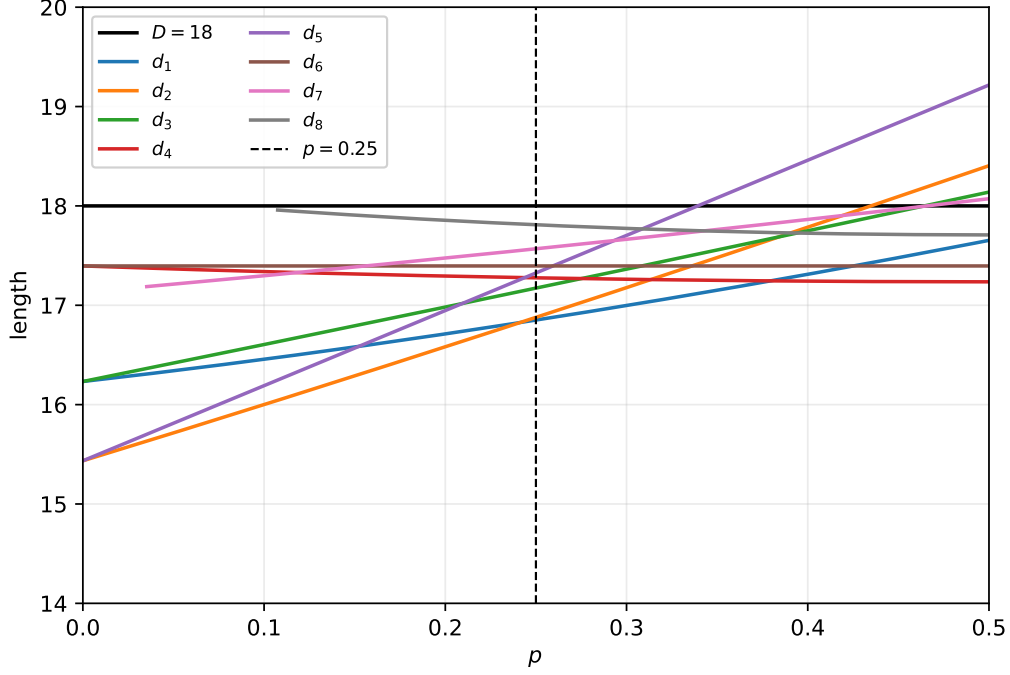


Figure 9: The eight classes on the pentagonal prism, $b = 4$, $k = 3.5$. The horizontal line is the obvious path $D = 18$; d_6 is horizontal because it does not depend on p . The curves d_7 and d_8 start at $p = 0.0351$ and $p = 0.1074$ respectively.

7 The cylinder

The cylinder was the question that originally resisted. It is treated here as the limit $n \rightarrow \infty$ of the regular n -prism, which turns out to produce the governing equation without any new geometry.

Let r be the radius, so that the base height is $b = 2r$; write $h = 2kr$ and $D = h + 2r = 2r(k+1)$. The ant is at distance $2rp$ from the rim, measured along a diameter, and likewise the honey.

7.1 The limit of the general formula

Setting $b = 1$, $w = \frac{1}{2} - p$, and writing $x = i\beta$, $y = j\beta$ for the rolling angles of the two bases, the general components of Section 8 become, as $n \rightarrow \infty$ (so that $\sigma \rightarrow 0$ and $\rho \rightarrow \frac{1}{2}$),

$$\Delta x = \frac{y - x}{2} - w(\sin x + \sin y), \quad \Delta y = (k + 1) - w(\cos y - \cos x). \quad (28)$$

Proposition 7.1 (Feasibility becomes stationarity). *In the limit $n \rightarrow \infty$ the two feasibility conditions of Section 8 coincide with the stationarity conditions for the length:*

$$\frac{\partial d^2}{\partial x} = -2F_1, \quad \frac{\partial d^2}{\partial y} = 2F_2,$$

where

$$F_1 = \Delta x \left(\frac{1}{2} + w \cos x \right) + \Delta y w \sin x, \quad F_2 = \Delta x \left(\frac{1}{2} - w \cos y \right) + \Delta y w \sin y.$$

This is as it should be. On a polygonal base the condition is discrete — the segment must cross a particular side — whereas on a smooth surface it becomes the condition that the length be stationary.

7.2 Where the symmetry comes from

In the original treatment the relation $x_1 = \pi - x$ between the two rolling angles was imposed as an observation. It is in fact forced:

Proposition 7.2 (Two branches).

$$F_1 - F_2 = 2w \cos \frac{x+y}{2} \left[\Delta x \cos \frac{x-y}{2} + \Delta y \sin \frac{x-y}{2} \right].$$

Hence every critical point lies on one of

$$(A) \quad x + y = \pi, \quad (B) \quad \tan \frac{x-y}{2} = -\frac{\Delta x}{\Delta y}.$$

Branch (A) is the symmetry $x_1 = \pi - x$. On it, $\sin y = \sin x$ and $\cos y = -\cos x$, so with $q := 1 - 2p = 2w$

$$\Delta x = \frac{\pi}{2} - x - q \sin x, \quad \Delta y = (k+1) + q \cos x,$$

and the two conditions collapse into one:

$$k q \sin x + \left(\frac{\pi}{2} - x \right) (1 + q \cos x) = 0, \quad \text{i.e.} \quad k = \frac{(x - \frac{\pi}{2})(\cos x + \frac{1}{1-2p})}{\sin x}. \quad (29)$$

Branch (B) also yields critical points, but in every case examined — $k \in \{1.5, 2, 3.5, 8\}$, $p \in \{0.1, 0.25, 0.4\}$ — none of them is shorter than the obvious path. This is why the cylinder has only one shorter path: not because other critical points fail to exist, but because they are all longer than D .

7.3 A closed parametric form

Equation (29) is transcendental in x , and there is no elementary expression for x in terms of k . The original treatment therefore solved it numerically and fed the resulting x into a second formula. That step can be avoided: k itself can be eliminated.

From (29), $\Delta y = -\Delta x \frac{1 + q \cos x}{q \sin x}$, whence

Theorem 7.3 (Parametric solution for the cylinder). *With $q = 1 - 2p$ and $b = 2r$, the shorter path on the cylinder is given parametrically by*

$$k(x) = \frac{(2x - \pi)(1 + q \cos x)}{2q \sin x}, \quad d(x) = 2r \frac{|\pi - 2x - 2q \sin x| \sqrt{q^2 + 2q \cos x + 1}}{2q |\sin x|}, \quad (30)$$

and consequently

$$\frac{d}{D}(x) = \frac{|\pi - 2x - 2q \sin x| \sqrt{q^2 + 2q \cos x + 1}}{|(2x - \pi)(1 + q \cos x) + 2q \sin x|}.$$

Everything is symbolic; only the parametrisation has changed from k to x . Choosing x gives a pair (k, d) directly, and questions such as “which k minimises d/D ” become one-dimensional problems in x .

7.4 The case $p = 0$

For $p = 0$ we have $q = 1$, and the term carrying k in (29) contains $\sin x$, so $x = \pi$ solves the equation for every k . Substituting gives $\Delta x = -\pi/2$ and $\Delta y = k$, hence

$$d_0 = 2r \sqrt{\frac{\pi^2}{4} + k^2} = r \sqrt{\pi^2 + 4k^2}. \quad (31)$$

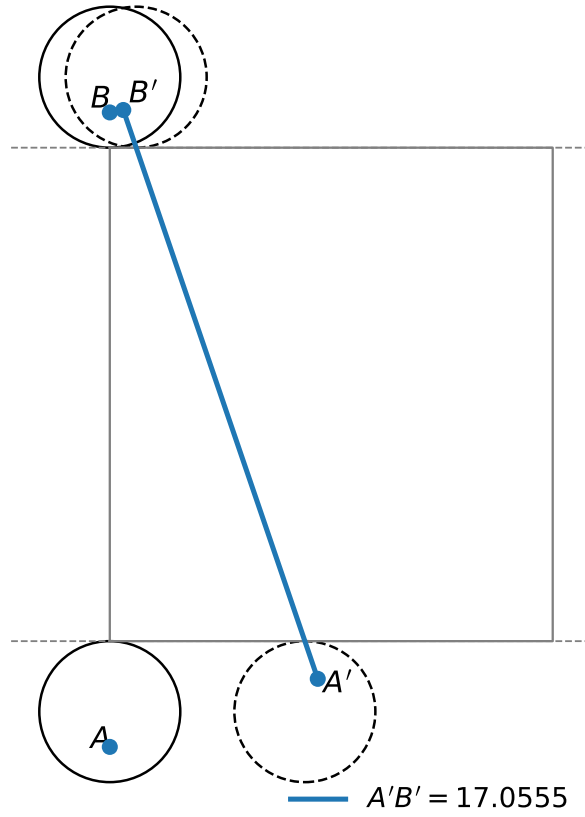


Figure 10: The cylinder unfolded, $b = 2r = 4$, $k = 3.5$, $p = 0.25$. The solid circles are the bases in their initial position, the dashed ones after rolling: the ant's (lower) base through $x = 2.76738$ and the honey's (upper) base through $y = \pi - x = 0.37421$ radians, their sum being π — the symmetry of branch (A). The straight segment $A'B'$ has length 17.0555 against $D = 18$.

This is the one value of p for which a closed form in k exists. The path is simply the one that wraps exactly half-way around the cylinder: horizontal extent πr , vertical extent h .

From (31), $d_0 < D = 2r(k+1)$ requires

$$k > \frac{\pi^2}{8} - \frac{1}{2} = 0.73370\dots,$$

and minimising d_0/D gives

$$k_{opt} = \frac{\pi^2}{4} = 2.4674\dots, \quad \frac{d}{D} = 0.8436\dots$$

7.5 Numerical values

p	$k_{\min}$	k_{opt}	d/D at k_{opt}
0	0.7337	2.467	0.8436
0.01	0.7482	2.497	0.8484
0.05	0.8019	2.623	0.8672
0.25	0.9541	3.564	0.9475
0.40	0.9932	5.652	0.9885

Here $k_{\min}$ is the smallest k for which a shorter path exists and k_{opt} the one giving the largest relative saving. As p increases the saving shrinks: at $p = 0.4$ the best possible gain is barely one per cent.

Example 7.4. For $r = 2$, $k = 3.5$, $p = 0.25$, solving (29) gives $x = 2.76738$, hence $y = \pi - x = 0.37421$, and $d = 17.05550$ against $D = 18$. In the convention of Section 8 the angle x belongs to the ant's base and y to the honey's; the mirror-equivalent path, in which the roles of the two angles are exchanged, arises from placing the ant at distance bp (rather than $b(1-p)$) from the unfolding rim and has the same length.

8 One formula for every prism

The four prisms treated above were computed one at a time. They need not be: a single pair of formulas gives the length and the direction of every candidate path on every regular prism, and the cylinder follows as the limit. This section states it.

8.1 Components

Theorem 8.1 (General components). *Let $n \geq 3$, $\beta = 2\pi/n$, and put*

$$\sigma = \frac{s}{b} = \begin{cases} 2 \tan \frac{\pi}{2n}, & n \text{ odd} \\ \tan \frac{\pi}{n}, & n \text{ even} \end{cases} \quad \rho = \frac{r}{b} = \begin{cases} \frac{\cos(\pi/n)}{1 + \cos(\pi/n)}, & n \text{ odd} \\ \frac{1}{2}, & n \text{ even}. \end{cases}$$

For the path obtained by rolling the lower base i steps and the upper base j steps, set $u = p - \rho$, $v = p - 1 + \rho$ and $m = j - i$. Then, in units of b ,

$$\frac{\Delta x}{b} = m\sigma + u \sin(j\beta) + v \sin(i\beta), \quad \frac{\Delta y}{b} = k + 2\rho + u \cos(j\beta) - v \cos(i\beta), \quad (32)$$

and $d = \sqrt{\Delta x^2 + \Delta y^2}$. Written out,

$$\begin{aligned} \frac{d^2}{b^2} = & m^2 \sigma^2 + (k + 2\rho)^2 + u^2 + v^2 - 2uv \cos((i+j)\beta) \\ & + 2m\sigma(u \sin(j\beta) + v \sin(i\beta)) + 2(k + 2\rho)(u \cos(j\beta) - v \cos(i\beta)). \end{aligned} \quad (33)$$

Proof. By Lemma 1.5 the centre of D_i is at $((i + \frac{1}{2})s, Y_b - r)$ and the map $D_0 \rightarrow D_i$ is a direct isometry with rotation angle $i\beta$. Since $Y_b = b$ and the centre of D_0 is at height $b - r = c$, the ant's position relative to that centre is $A_0(p) - O_A(0) = (0, bp - c)$, which is independent of the rolling. Hence

$$A_i(p) = O_A(i) + \rho_{i\beta} \begin{pmatrix} 0 \\ bp - c \end{pmatrix}, \quad B_j(p) = O_B(j) + \rho_{-j\beta} \begin{pmatrix} 0 \\ bp - r \end{pmatrix},$$

where ρ_θ is rotation by θ , so $\rho_\theta(0, \mu) = (-\mu \sin \theta, \mu \cos \theta)$. Subtracting and dividing by b gives (32); expanding the squares and using $\cos(i\beta)\cos(j\beta) - \sin(i\beta)\sin(j\beta) = \cos((i+j)\beta)$ gives (33). $\square$

Formula (32) reproduces every formula in Sections 3–6. It was checked symbolically against all eleven length formulas obtained by hand for $n = 3, 4, 6$ and against d_6 for $n = 5$, and numerically against the constructed geometry on 4802 shorter paths for $n = 3, \dots, 15$, with largest discrepancy $4.3 \cdot 10^{-14}$.

8.2 Angles

Definition 8.2 (Angle convention). The angle at which the ant sets off is measured in the frame of its own base, that is, rotated back by $i\beta$:

$$\theta = i\beta - \alpha, \quad \alpha = \text{atan2}(\Delta y, \Delta x).$$

Since $\Delta y > 0$ always, $\alpha \in (0, \pi)$ and the atan2 resolves into

$$\theta = \begin{cases} \frac{2i\pi}{n} - \arctan \frac{\Delta y}{\Delta x}, & \Delta x > 0, \\ \arctan \frac{\Delta y}{|\Delta x|} + \frac{(2i-n)\pi}{n}, & \Delta x < 0, \\ \frac{2i\pi}{n} - \frac{\pi}{2}, & \Delta x = 0. \end{cases} \quad (34)$$

The offset in the middle line, $i\beta - \pi = (2i - n)\pi/n$, accounts for every constant appearing in the angle formulas of the earlier sections: $-\pi/3$ for $n = 3$ with $i = 1$, zero for $n = 4$ with $i = 2$ and for $n = 6$ with $i = 3$, $-\pi/3$ for $n = 6$ with $i = 2$, and $-\pi/5$ for $n = 5$ with $i = 2$. The formulas of the “constant minus arctan” shape are the case $\Delta x > 0$, where the constant is $2i\pi/n$. All twelve hand-derived angle formulas were verified against (34) over the whole range $k \in [2, 6]$, $p \in [0, 0.45]$, with maximum discrepancy $2.8 \cdot 10^{-14}$.

8.3 Feasibility

A straight segment in the unfolded picture is a path on the prism only if it stays inside the material. The criterion is exact and requires no sampling.

Lemma 8.3 (Feasibility criterion). *Let $A \in D_i$, $B \in G_j$, and let the segment $[A, B]$ meet the lines $y = Y_b$ and $y = Y_t$ at $P = (x_P, Y_b)$ and $Q = (x_Q, Y_t)$. Then*

$$[A, B] \subseteq D_i \cup \Pi \cup G_j \iff x_P \in [is, (i+1)s] \text{ and } x_Q \in [js, (j+1)s].$$

Proof. D_i is convex, lies in $y \leq Y_b$, and meets the line $y = Y_b$ exactly in its contact side $[is, (i+1)s]$ (Lemma 1.5). If x_P lies in that interval then $P \in D_i$, and convexity gives $[A, P] \subseteq D_i$; similarly for $[Q, B]$ and G_j . The middle portion lies in the strip, every point of which is a surface point, since the strip is the development of the lateral surface and may be extended indefinitely. Conversely, if the segment lies in the union then P , having y -coordinate Y_b and being approached from below, must lie in D_i , hence in the contact side. $\square$

In closed form: with

$$\alpha = \rho - v \cos(i\beta), \quad \gamma = \rho + u \cos(j\beta), \quad \Delta y = k + \alpha + \gamma,$$

and $\hat{x}_A = \frac{\sigma}{2} - v \sin(i\beta)$, $\hat{x}_B = (m + \frac{1}{2})\sigma + u \sin(j\beta)$ (both measured from is), the criterion reads

$$0 \leq \hat{x}_A + \frac{\alpha}{\Delta y} \Delta x \leq \sigma, \quad m\sigma \leq \hat{x}_B - \frac{\gamma}{\Delta y} \Delta x \leq (m+1)\sigma. \quad (35)$$

Remark 8.4. Sampling the segment at finitely many points is not a reliable substitute. Near the boundary of feasibility the violation is of order 10^{-5} and is easily missed; and when $p = 0$ the ant sits exactly on the boundary of its base, where point-in-polygon routines are unreliable and may reject a perfectly valid path. In a systematic comparison over 12 525 configurations, a sampling test disagreed with (35) in 639 cases: it wrongly rejected 565 valid paths (all at $p = 0$) and wrongly accepted 74 invalid ones.

8.4 Thresholds in p

Proposition 8.5 (Feasibility as a quadratic in p). *Write*

$$G_P(p) = \left(\frac{\sigma}{2} - v \sin(i\beta)\right) \Delta y + \alpha \Delta x, \quad G_Q(p) = \left(\frac{\sigma}{2} + u \sin(j\beta)\right) \Delta y - \gamma \Delta x.$$

Then the path is feasible exactly when

$$G_P \geq 0, \quad \sigma \Delta y - G_P \geq 0, \quad G_Q \geq 0, \quad \sigma \Delta y - G_Q \geq 0,$$

each of which is a quadratic function of p with the same leading coefficient

$$[p^2] G_P = [p^2] G_Q = -\sin((i+j)\beta).$$

In particular the conditions degenerate to linear ones exactly when $n \mid 2(i+j)$.

Proof. Multiplying (35) by $\Delta y > 0$ gives the four stated inequalities. Each of $u, v, \Delta x, \alpha, \gamma, \Delta y$ is affine in p , so each product is quadratic. For the leading coefficient, differentiate: $dv/dp = 1$, $d\alpha/dp = -\cos(i\beta)$, $d\Delta y/dp = \cos(j\beta) - \cos(i\beta)$ and $d\Delta x/dp = \sin(i\beta) + \sin(j\beta)$, whence

$$[p^2] G_P = -\sin(i\beta)(\cos(j\beta) - \cos(i\beta)) - \cos(i\beta)(\sin(j\beta) + \sin(i\beta)) = -\sin((i+j)\beta),$$

and the same computation for G_Q . □

Thresholds p_{min} are therefore roots of ordinary quadratics, obtainable in closed form; the hexagonal and pentagonal examples of Sections 5.2 and 6 are instances.

8.5 Odd and even prisms

Two consequences of (32) deserve to be separated out, because they are the structural core of the whole subject. Call a class p -independent if its length does not depend on p .

Theorem 8.6 (Even prisms). *Let n be even. Then every p -independent class satisfies $m = j - i \in 2\mathbb{Z}$ and*

$$d = b\sqrt{(m\sigma)^2 + (k+1)^2} \geq D,$$

with equality if and only if $i = j$. No even prism has a p -independent path shorter than the obvious one, for any $k > 0$, any p , and including helical paths.

Proof. For even n we have $\rho = \frac{1}{2}$, so $u = v$ and, by Definition 1.2, $\varepsilon = 0$. A short computation from (32) (carried out in full in Appendix A) then gives $\Delta x = m\sigma b$ and $\Delta y = D$ for every p -independent pair, whence the claim. □

Theorem 8.7 (Odd prisms). *Let $n \geq 3$ be odd. Then for every $k > 0$ and every $p \in [0, \frac{1}{2}]$ the pair $(\frac{n-1}{2}, \frac{n+1}{2})$ gives a p -independent, geometrically feasible, non-helical path of length*

$$d = b\sqrt{k^2 + 2k \cos \frac{\pi}{n} + 1} < D.$$

The proof is Appendix A. The mechanism is the asymmetry $\varepsilon = R - r$ of Definition 1.2: it is the only term in (32) capable of reducing the vertical component below D , and it is positive precisely for odd n .

Remark 8.8 (What is not general). Formula (32) gives exact values for every n . What it cannot always give is a *tidy* closed form. Expressions in square roots, of the kind obtained for $n = 3, 4, 5, 6$, exist only when the regular n -gon is constructible with ruler and compass, i.e. for

$$n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, \dots$$

For $n = 7$, $\cos(\pi/7)$ is a root of an irreducible cubic with three real roots (*casus irreducibilis*) and cannot be written using square roots alone. The formula still gives the exact value; the coefficients simply remain in the form $\cos(\pi/7)$, $\sin(3\pi/14)$ and so on. The same applies to $n = 9, 11, 13, 14, 18, 19, \dots$

Example 8.9 (The heptagonal prism). At $b = 4$, $k = 3.5$, $p = 0.25$ the prism with $n = 7$ has thirteen classes of shorter paths. The class with index pair $(3, 4)$ is the only one whose components do not involve p ; by Theorem 8.7 its length is $b\sqrt{k^2 + 2k \cos(\pi/7) + 1}$, and indeed $2 \cos(\pi/7) = 1.801938 \dots$

9 When is there no shorter path at all?

The square prism requires $k > 3/2$ before any saving exists, even with the ant on the rim (Section 3); the triangular prism requires nothing at all (Remark 4.2). This section settles the question for every n .

9.1 Odd prisms

By Theorem 8.7 the p -independent path of length $b\sqrt{k^2 + 2k \cos(\pi/n) + 1}$ is strictly shorter than D for every $k > 0$ and every p . Odd prisms therefore never have a height threshold.

9.2 Even prisms

Here p -independent paths are never shorter (Theorem 8.6), so the threshold is set by some other class. With the ant on the rim, $p = 0$, the best is the pair $(i, j) = (n/2, 0)$, for which (32) gives

$$\Delta x = -\frac{n}{2} b \tan \frac{\pi}{n}, \quad \Delta y = h.$$

Since $b = s \cot(\pi/n)$ for even n , we have $\frac{n}{2} b \tan(\pi/n) = \frac{ns}{2} = \frac{P}{2}$, half the perimeter of the base. So this is the path that wraps exactly half-way around the lateral surface, from a rim point to the diametrically opposite rim point, and

$$d = \sqrt{\left(\frac{P}{2}\right)^2 + h^2},$$

the direct analogue of the cylinder's $d_0 = \sqrt{(\pi r)^2 + h^2}$ in (31).

The condition $d < D = h + b$ becomes $\left(\frac{P}{2}\right)^2 < 2hb + b^2$, that is:

Theorem 9.1 (Height threshold for even prisms). *For even n a shorter path exists if and only if*

$$k > k_{\min}(n) = \frac{n^2 \tan^2(\pi/n)}{8} - \frac{1}{2}.$$

n	$k_{\min}(n)$	exact	shorter path at $k = 1$?
4	1.5	3/2	no
6	1.0	1	borderline: $d = D$
8	0.87258		yes
10	0.81966		yes
12	0.79234		yes
20	0.75428		yes
50	0.73696		yes
∞	0.73370	$\pi^2/8 - \frac{1}{2}$	yes (the cylinder)

Since $n \tan(\pi/n)$ decreases to π , the threshold decreases monotonically to the cylinder value $\pi^2/8 - \frac{1}{2}$ found in Section 7. The condition $k_{\min}(n) \leq 1$ is equivalent to

$$n \tan \frac{\pi}{n} \leq 2\sqrt{3} = 3.4641 \dots,$$

which fails for $n = 4$ (left side 4), holds with equality for $n = 6$ (left side $6 \tan 30^\circ = 2\sqrt{3}$), and holds strictly from $n = 8$ on. So the square prism is the only prism admitting no shorter path at $k = 1$, and the hexagonal one sits exactly on the boundary. Allowing helical paths does not lower any of these thresholds, as one would expect — a larger $|m|$ means a larger $|\Delta x|$. The “only if” direction of Theorem 9.1 rests on the pair $(n/2, 0)$ at $p = 0$ being optimal; a direct search over all pairs, all $p \in [0, \frac{1}{2}]$ and all $|m|$, helical paths included, confirms that just below $k_{\min}(n)$ no feasible path is shorter than D for any p (checked for $n = 4, 6, 8$).

9.3 Thresholds in p : which prisms are affected

Section 5.2 showed that on the hexagonal prism two of the four classes do not exist for small p . The phenomenon is the rule rather than the exception. Counting pairs (i, j) that are shorter than D somewhere on $p \in [0, \frac{1}{2}]$, and among them those that are not feasible at $p = 0$:

n	$k = 2$		$k = 3.5$		$k = 8$	
	total	with threshold	total	with threshold	total	with threshold
3	8	0	10	0	10	0
4	6	0	6	0	6	0
5	18	2	18	6	22	14
6	12	0	12	6	16	14
7	28	14	30	22	34	26
8	20	14	20	14	26	24
10	30	24	30	28	38	36
12	42	36	42	40	50	48

Two regularities:

- The triangular and square prisms are the exceptions: no class that is shorter than D has a feasibility threshold, for any k . (The triangular prism does have pairs with a threshold — $(0, 3)$ from $p = 0.25$ at $k = 3.5$, for instance — but those paths are never shorter than D , so they do not appear on a graph like Figure 9.)

- From $n = 5$ on the phenomenon occurs for both parities, and the proportion of affected classes grows with n and with k ; at $n = 12$, $k = 8$ it is 48 out of 50.

The count includes classes that are shorter than D only in a band near $p = \frac{1}{2}$ — the analogues of d_5 on the triangular prism (Section 4.3). From $n = 5$ on, such a class places its point far from the centre of the base at $p = 0$ and is infeasible there, so it contributes to the “with threshold” column; on the triangular prism itself the analogous class $(0, 1)$ is feasible for every p , which is why the row $n = 3$ stays at zero. For the pentagonal prism — the smallest odd one affected — the classes concerned at $k = 3.5$ are d_7 and d_8 , whose thresholds appear only for $k > 3$ and $k > 2$ respectively (Section 6), together with the pair $(1, 2)$ and its mirror $(4, 3)$, a class of exactly this latter kind; at $k = 2$ that pair is the only one.

10 Three conjectures

The sections above describe what happens for a given prism. Three questions remain about how the answers behave as the prism changes. They were originally posed as conjectures; all three are settled in the appendices, though not all of them in the way expected.

Throughout, “shorter path” means a class of paths of equal length, shorter than the obvious path D .

H1: the p -independent path

H1. Every odd regular prism has at least one p -independent shorter path, that is, a shorter path whose length does not depend on the distance of A and B from the rim, while even prisms have none. The shortest such path has length

$$d = b\sqrt{k^2 + 2k \cos \frac{\pi}{n} + 1}.$$

This is true, and Appendix A gives a complete proof. One point of formulation needs care: even prisms do possess p -independent paths — infinitely many classes of them — but none is shorter than the obvious path. The correct statement is Theorem 8.6 together with Theorem 8.7.

The appendix also settles uniqueness, which does not hold unconditionally: the path above is the only shorter p -independent one while $k < k^*(n)$, where $k^*(n) \geq 6$ for every odd n , with equality exactly at $n = 3$. Beyond that threshold further classes appear; for $n = 5$ a second one at $k > 22.47$.

H2: many prisms, bounded height

H2. The number of shorter paths is unbounded as n grows while k stays bounded, even if only n rollings of the two bases are allowed — that is, without helical paths.

This is false. Appendix B shows that for every fixed $k \neq 1$ the number of classes saturates: it stops changing at around $n \approx 100$ and remains constant thereafter — 19 classes at $k = 2$, 11 at $k = 3.5$, 9 at $k = 5$ (all at $p = 0.25$).

There is exactly one exception, and it is sharp. At $k = 1$, for every p , the number grows without bound, roughly like $\sqrt{n}$. The reason is a degeneracy: the two curves that describe feasibility in the limit $n \rightarrow \infty$ become tangent precisely when $k = 1$, and the determinant measuring their transversality is $-(k - 1)(2p - 1)$.

H3: one prism, unbounded height

H3. The number of shorter paths is unbounded, for every n , as k grows without bound — but only if an unlimited number of rollings of the two bases is allowed, that is, if helical paths are admitted.

Both halves are true. Appendix C proves the first and quantifies it: the number of paths is asymptotically $C(n, p)\sqrt{k}$ with an explicit constant, which agrees with measurement to within 0.4%. The second half is confirmed by the counts: restricted to $0 \leq i, j \leq n$, the number saturates and does not move between $k = 10^3$ and $k = 10^5$.

A remark on the two

H2 and H3 turn on the same ratio: the width σ of the contact side against the range that has to be covered. In H2 the number of sides grows, $\sigma \sim 1/n$ shrinks, and the feasibility conditions tighten from inequalities into equations — which is why H2 fails. In H3 the prism is fixed, σ stays put, the conditions remain inequalities, and it is the range of m that grows — which is why H3 holds.

A Proof of H1

This appendix proves that every odd regular prism has a p -independent path strictly shorter than the obvious one, that no even prism has, and settles the question of uniqueness. Notation is that of Sections 1 and 8; in particular $\beta = 2\pi/n$, $c = b - r$, $\varepsilon = c - r = b - 2r$, and $D = h + b$.

A.1 The criterion for p -independence

Definition A.1. For $i, j \in \mathbb{Z}$ put $d_{ij}(p) = \|B_j(p) - A_i(p)\|$. The pair (i, j) is p -independent if d_{ij} is constant on $[0, \frac{1}{2}]$.

By the proof of Theorem 8.1, the rolling isometries do not depend on p while $A_0(p)$ and $B_0(p)$ are affine in p ; hence A_i and B_j are affine in p with constant velocity vectors

$$A'_i(p) = \begin{pmatrix} -b \sin(i\beta) \\ b \cos(i\beta) \end{pmatrix}, \quad B'_j(p) = \begin{pmatrix} b \sin(j\beta) \\ b \cos(j\beta) \end{pmatrix}. \quad (36)$$

Lemma A.2. (i, j) is p -independent if and only if $A'_i(p) = B'_j(p)$.

Proof. Write $v(p) = B_j(p) - A_i(p) = v_0 + p v_1$ with $v_1 = B'_j - A'_i$ constant. Then

$$\|v(p)\|^2 = \|v_0\|^2 + 2p(v_0 \cdot v_1) + p^2\|v_1\|^2,$$

a quadratic in p . It is constant on an interval of positive length if and only if the coefficients of p^2 and p vanish, i.e. if and only if $\|v_1\|^2 = 0$, that is $v_1 = 0$. The converse is immediate. $\square$

Lemma A.3. (i, j) is p -independent if and only if $i + j \equiv 0 \pmod{n}$.

Proof. By Lemma A.2 and (36) the condition reads $-\sin(i\beta) = \sin(j\beta)$ and $\cos(i\beta) = \cos(j\beta)$, i.e. $e^{ij\beta} = e^{-ij\beta}$, i.e. $(i + j)\beta \in 2\pi\mathbb{Z}$. Since $\beta = 2\pi/n$ this is $n \mid i + j$. $\square$

Remark A.4. The condition includes $i + j = 0$, that is the pair $(0, 0)$: the obvious path itself is p -independent, of length exactly D . This trivial class must not be overlooked when counting.

A.2 The length of a p -independent path

Theorem A.5. *Let $i + j \equiv 0 \pmod{n}$, and put $\theta = i\beta$, $m = j - i$. Then the vector $v = B_j(p) - A_i(p)$ is constant, with*

$$\Delta x = ms - \varepsilon \sin \theta, \quad \Delta y = D - \varepsilon(1 - \cos \theta), \quad (37)$$

so that $d_{ij} = \sqrt{\Delta x^2 + \Delta y^2}$.

Proof. From $n \mid i + j$ we get $j\beta \equiv -i\beta$, so $\sin(j\beta) = -\sin \theta$ and $\cos(j\beta) = \cos \theta$. Substituting into the expressions of Theorem 8.1, written in absolute lengths,

$$\begin{aligned} \Delta x &= ms + (bp - r) \sin(j\beta) + (bp - c) \sin \theta = ms + \sin \theta [(bp - c) - (bp - r)] = ms - \varepsilon \sin \theta, \\ \Delta y &= h + 2r + (bp - r) \cos \theta - (bp - c) \cos \theta = h + 2r + \varepsilon \cos \theta. \end{aligned}$$

Since $b = 2r + \varepsilon$, we have $h + 2r = h + b - \varepsilon = D - \varepsilon$, which gives the second formula. The parameter p has cancelled, which incidentally reproves the “if” direction of Lemma A.3. $\square$

A.3 Even prisms

Theorem A.6 (Theorem 8.6, proved). *Let n be even. Then every p -independent pair has $m = j - i$ even and*

$$d_{ij} = \sqrt{(ms)^2 + D^2} \geq D,$$

with equality if and only if $i = j$. No even prism has a p -independent path shorter than D , for any $k > 0$, any p , including helical paths.

Proof. For even n , Definition 1.2 gives $\varepsilon = 0$, so (37) reduces to $\Delta x = ms$, $\Delta y = D$, independently of θ . Hence $d_{ij}^2 = (ms)^2 + D^2 \geq D^2$ with equality exactly when $m = 0$. For the parity: n even and $i + j = Mn$ force $i + j$ even, hence $m = (i + j) - 2i$ even. $\square$

Theorem A.6 is both stronger and simpler than the assertion that even prisms have no p -independent path. They have many; none is shorter. For instance the pair $(2, 2)$ on the square prism is p -independent, feasible and non-helical, of length exactly D .

A.4 Odd prisms: existence

Throughout this subsection $n \geq 3$ is odd and

$$C = \cos \frac{\pi}{n}, \quad S = \sin \frac{\pi}{n}, \quad \varphi = \frac{\pi}{2n}, \quad \tau = \tan \varphi,$$

so that, from Definitions 1.1 and 1.2,

$$s = 2b\tau, \quad \varepsilon = b\tau^2, \quad r = RC, \quad b = R(1 + C), \quad RS = \frac{s}{2}. \quad (38)$$

Theorem A.7 (Theorem 8.7, proved). *Let $n \geq 3$ be odd and $i = \frac{n-1}{2}$, $j = \frac{n+1}{2}$, so $i + j = n$ and $m = 1$. Then for every $k > 0$ and every $p \in [0, \frac{1}{2}]$:*

- (i) *the pair is p -independent;*
- (ii) *the segment is geometrically feasible;*
- (iii) *its length is $d = b\sqrt{k^2 + 2k \cos(\pi/n) + 1}$;*
- (iv) *$d < D$ strictly.*

Proof. (i) Lemma A.3, since $i + j = n$.

(iii) Here $\theta = i\beta = \pi - \pi/n$, so $\sin \theta = S$ and $\cos \theta = -C$, and (37) gives $\Delta x = s - \varepsilon S$ and $\Delta y = D - \varepsilon(1 + C)$. By (38),

$$\varepsilon S = b\tau^2 \cdot 2 \sin \varphi \cos \varphi = 2b\tau \sin^2 \varphi = s \sin^2 \varphi = \frac{s}{2}(1 - C),$$

so $\Delta x = s - \frac{s}{2}(1 - C) = \frac{s}{2}(1 + C) = b\tau \cdot 2 \cos^2 \varphi = 2b \sin \varphi \cos \varphi = bS$. Similarly $\varepsilon(1 + C) = 2b \sin^2 \varphi = b(1 - C)$, so $\Delta y = D - b(1 - C) = h + bC$. Hence

$$d^2 = b^2 S^2 + (h + bC)^2 = b^2 + h^2 + 2hbC,$$

and substituting $h = kb$ gives (iii).

(iv) From the same computation, $d^2 - D^2 = b^2 + h^2 + 2hbC - (h + b)^2 = 2hb(C - 1) < 0$, since $h, b > 0$ and $C < 1$ for $n \geq 3$.

(ii) We apply Lemma 8.3. Translate so that the contact side of D_i is $[0, s]$; since $j = i + 1$, that of G_j is $[s, 2s]$. From the coordinates in the proof of Theorem 8.1, $y_A = c + (bp - c) \cos \theta = R - (bp - R)C$, so using $r = RC$ and $b = R(1 + C)$,

$$Y_b - y_A = b - R + (bp - R)C = r + (bp - R)C = RC + bpC - RC = bpC,$$

and likewise, using $r(1 + C) = RC(1 + C) = bC$,

$$y_B - Y_t = r - (bp - r)C = r(1 + C) - bpC = bC(1 - p).$$

These sum to bC , consistent with $\Delta y = h + bC$. Put

$$u := \frac{bC}{h + bC} = \frac{C}{k + C} \in (0, 1),$$

so the segment reaches $y = Y_b$ at parameter $t_1 = pu$ and $y = Y_t$ at t_2 with $1 - t_2 = (1 - p)u$. For the horizontal coordinates, $x_A = \frac{s}{2} - (bp - R)S = s - bpS$ using $RS = \frac{s}{2}$, and $x_B = \frac{3s}{2} - (bp - r)S = \frac{3s}{2} + \frac{s}{2}C - bpS$ using $rS = RCS = \frac{s}{2}C$. (Check: $x_B - x_A = \frac{s}{2}(1 + C) = bS = \Delta x$.) Therefore

$$x_P = x_A + t_1 \Delta x = s - bpS(1 - u).$$

Since $0 \leq p \leq \frac{1}{2}$, $0 < 1 - u < 1$ and $bS = \frac{s}{2}(1 + C) < s$, we get $\frac{s}{2} < s - \frac{bS}{2} \leq x_P \leq s$, so $x_P \in [0, s]$. For the other end,

$$x_Q = x_B - (1 - p)u \Delta x = s + bS[1 - p - (1 - p)u],$$

so $x_Q - s = bS(1 - p)(1 - u) > 0$ and $2s - x_Q = s - bS(1 - p)(1 - u) > s - bS = \frac{s}{2}(1 - C) > 0$. Hence $x_Q \in (s, 2s)$, and Lemma 8.3 applies. $\square$

Remark A.8. The proof of (ii) shows that *no lower bound on k is needed*: the path is feasible for every $k > 0$, every $p \in [0, \frac{1}{2}]$ and every odd $n \geq 3$.

A.5 Classification of all p -independent paths

Lemma A.9. *Let n be odd and $n \mid i + j$. Then $\theta = i\beta$ is determined by $m = j - i$:*

$$\theta \equiv \begin{cases} \pi - \frac{m\pi}{n} \pmod{2\pi}, & m \text{ odd}, \\ -\frac{m\pi}{n} \pmod{2\pi}, & m \text{ even}. \end{cases}$$

Consequently d_{ij} depends only on m . Moreover (i, j) and $(i + n, j + n)$ give the same path on the prism.

Proof. From $i + j = Mn$ and $j - i = m$ we get $2i \equiv -m \pmod{n}$; as n is odd, 2 is invertible modulo n with $2^{-1} \equiv \frac{n+1}{2}$, so $i \equiv -m\frac{n+1}{2}$ and

$$\theta = \frac{2\pi i}{n} \equiv -\frac{m\pi(n+1)}{n} = -m\pi - \frac{m\pi}{n} \pmod{2\pi},$$

which gives the two cases. The length depends only on θ and m by (37), hence only on m . Finally $A_{i+n} = A_i + (ns, 0)$ and $B_{j+n} = B_j + (ns, 0)$, and translation by $(ns, 0)$ is the deck transformation of the development of the lateral surface, hence induces the identity on the prism. $\square$

Remark A.10 (Winding). It follows that classes are indexed by m , not by $M = (i+j)/n$: passing from (i, j) to $(i+n, j+n)$ leaves m , and therefore Δx and the winding number, unchanged. Paths with $|m| \geq n$ wind at least once around the prism and may be self-intersecting on the surface; neither affects the length nor Lemma 8.3.

Theorem A.11 (Length of class m). *Let n be odd, $m \in \mathbb{Z}$, and write $S_m = \sin(m\varphi)$, $C_m = \cos(m\varphi)$. Then $d_m = \sqrt{\Delta x^2 + \Delta y^2}$ with*

$$\begin{aligned} m \text{ odd} : \quad \Delta x &= 2b\tau(m - \tau S_m C_m), & \Delta y &= D - 2b\tau^2 C_m^2, \\ m \text{ even} : \quad \Delta x &= 2b\tau(m + \tau S_m C_m), & \Delta y &= D - 2b\tau^2 S_m^2. \end{aligned}$$

Classes m and $-m$ have equal length.

Proof. Substitute θ from Lemma A.9 into (37), using $s = 2b\tau$, $\varepsilon = b\tau^2$, $\sin(m\pi/n) = 2S_m C_m$, $1 + \cos(m\pi/n) = 2C_m^2$ and $1 - \cos(m\pi/n) = 2S_m^2$. Replacing m by $-m$ changes the sign of Δx and leaves Δy unchanged. $\square$

A.6 Which classes are shorter than D

Theorem A.12 (Threshold k^*). *Write $\Delta y = D - w$ with $w \geq 0$ as in Theorem A.11. Then*

$$d_m < D \iff w > 0 \text{ and } k > k^*(m, n),$$

where

$$k^*(m, n) = \begin{cases} \frac{(m - \tau S_m C_m)^2}{C_m^2} + \tau^2 C_m^2 - 1, & m \text{ odd}, C_m \neq 0, \\ \frac{(m + \tau S_m C_m)^2}{S_m^2} + \tau^2 S_m^2 - 1, & m \text{ even}, S_m \neq 0. \end{cases} \quad (39)$$

Moreover $w = 0$ if and only if $n \mid m$, in which case $\Delta x = ms$, $\Delta y = D$ and

$$d_m = \sqrt{(ms)^2 + D^2} \geq D \quad \text{for every } k > 0, \quad (40)$$

with equality if and only if $m = 0$.

Proof. With $0 \leq w < D$,

$$d_m^2 < D^2 \iff \Delta x^2 < D^2 - (D - w)^2 = w(2D - w) \iff \Delta x^2 + w^2 < 2Dw.$$

If $w = 0$ the right side is 0 and the inequality is impossible, so $d_m \geq D$.

For the characterisation of $w = 0$: for odd m , $w = 2b\tau^2 C_m^2$ vanishes iff $\cos(m\varphi) = 0$, i.e. $m \equiv n \pmod{2n}$ (an odd multiple of n); for even m , $w = 2b\tau^2 S_m^2$ vanishes iff $\sin(m\varphi) = 0$, i.e. $2n \mid m$. The union of these two sets is exactly the set of multiples of n . In that case $m\pi/n$ is a multiple of π , so $\sin \theta = 0$ and $\Delta x = ms$, while $\Delta y = D - w = D$; this gives (40).

For $w > 0$, dividing by $2w$ gives $D > \frac{\Delta x^2 + w^2}{2w}$, and since $D = b(k+1)$ this is $k > \frac{\Delta x^2 + w^2}{2bw} - 1$. Substituting the expressions of Theorem A.11, for odd m ,

$$\frac{\Delta x^2 + w^2}{2bw} = \frac{4b^2\tau^2(m - \tau S_m C_m)^2 + 4b^2\tau^4 C_m^4}{4b^2\tau^2 C_m^2} = \frac{(m - \tau S_m C_m)^2}{C_m^2} + \tau^2 C_m^2,$$

and analogously for even m . $\square$

Corollary A.13 (The case $m = 1$). $k^*(1, n) = 0$ for every odd $n \geq 3$.

Proof. With $S_1 = \sin \varphi$, $C_1 = \cos \varphi$, $\tau = \tan \varphi$ we have $\tau S_1 C_1 = \sin^2 \varphi$, so $1 - \tau S_1 C_1 = \cos^2 \varphi = C_1^2$ and

$$k^*(1, n) + 1 = \frac{(C_1^2)^2}{C_1^2} + \tau^2 C_1^2 = \cos^2 \varphi + \sin^2 \varphi = 1. \quad \square$$

Corollary A.13 is an independent proof of part (iv) of Theorem A.7.

A.7 Uniqueness

Theorem A.14 (Conditional uniqueness). *Let $n \geq 3$ be odd and put $k^*(n) = \min_{m \geq 2} k^*(m, n)$. For $0 < k < k^*(n)$ the classes $m = \pm 1$ give the only p -independent path strictly shorter than D ; for $k > k^*(n)$ at least one further class does. Moreover*

$$k^*(m, n) \geq 6 \quad \text{for all } m \geq 2,$$

with equality if and only if $(n, m) = (3, 2)$. In particular $k^*(n) \geq 6$ for every odd n , so uniqueness holds unconditionally for $0 < k < 6$.

Proof. The first two statements follow from Theorem A.12, Corollary A.13 and Lemma A.9, together with $d_m = d_{-m}$ and $d_0 = D$.

For the bound, let $m \geq 2$ and $n \nmid m$ (otherwise $w = 0$ and the class is never shorter, by (40)). From

$$\tau S_m C_m = \frac{\tau}{2} \sin \frac{m\pi}{n} \implies |\tau S_m C_m| \leq \frac{\tau}{2}, \quad \tau = \tan \frac{\pi}{2n} \leq \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} < 0.5774,$$

we distinguish three cases.

(1) m odd, hence $m \geq 3$. Since $C_m^2 \leq 1$ and $m - \tau S_m C_m \geq m - \frac{\tau}{2} > 0$,

$$k^*(m, n) + 1 \geq \left(m - \frac{\tau}{2}\right)^2 \geq (3 - 0.2887)^2 > 7.35.$$

(2) m even, $m \geq 4$. Likewise, using $S_m^2 \leq 1$,

$$k^*(m, n) + 1 \geq \left(m - \frac{\tau}{2}\right)^2 \geq (4 - 0.2887)^2 > 13.7.$$

(3) $m = 2$. Here a crude estimate is insufficient and an exact computation is required. Put $u := \cos \frac{\pi}{n}$; for odd $n \geq 3$ we have $u \in [\frac{1}{2}, 1)$, with $u = \frac{1}{2}$ exactly for $n = 3$. With $\varphi = \pi/2n$ we have $S_2 = \sin 2\varphi$, $C_2 = \cos 2\varphi = u$, $S_2^2 = 1 - u^2$, and

$$\tau S_2 C_2 = \frac{\sin \varphi}{\cos \varphi} \cdot 2 \sin \varphi \cos \varphi \cdot \cos 2\varphi = 2 \sin^2 \varphi \cdot u = (1 - u)u, \quad \tau^2 S_2^2 = \frac{1 - u}{1 + u} (1 - u^2) = (1 - u)^2.$$

Hence

$$k^*(2, n) + 1 = \frac{(2 + u - u^2)^2}{1 - u^2} + (1 - u)^2.$$

We claim this is ≥ 7 . Multiplying by $1 - u^2 > 0$, the claim is equivalent to

$$(2 + u - u^2)^2 + (1 - u)^2(1 - u^2) \geq 7(1 - u^2).$$

The left-hand side equals $(u^4 - 2u^3 - 3u^2 + 4u + 4) + (1 - 2u + 2u^3 - u^4) = -3u^2 + 2u + 5$, so the claim becomes

$$-3u^2 + 2u + 5 \geq 7 - 7u^2 \iff 4u^2 + 2u - 2 \geq 0 \iff (2u - 1)(u + 1) \geq 0,$$

which holds exactly for $u \geq \frac{1}{2}$, with equality only at $u = \frac{1}{2}$. Thus $k^*(2, n) \geq 6$, with equality if and only if $n = 3$.

In all three cases $k^*(m, n) \geq 6$, and equality occurs only in case (3) with $n = 3$, i.e. for $(n, m) = (3, 2)$. Finally the minimum is attained, since $k^*(m, n) \rightarrow \infty$ as $m \rightarrow \infty$. $\square$

Remark A.15 (Exact values). Formula (39) is directly computable:

n	3	5	7	9	11	15	101
$k^*(n)$	6	12.472	12.684	10.420	9.504	8.756	8.016
attained at	$m = 2$	$m = 2$	$m = 3$	$m = 3$	$m = 3$	$m = 3$	$m = 3$

The value $k^*(3) = 6$ is exact. The branch $m = 2$ increases with n ; the branch $m = 3$ decreases, with limit

$$\lim_{n \rightarrow \infty} k^*(3, n) = \lim_{\varphi \rightarrow 0} \left[\frac{(3 - \tau S_3 C_3)^2}{C_3^2} + \tau^2 C_3^2 - 1 \right] = 9 - 1 = 8,$$

since $\tau \rightarrow 0$ and $C_3 = \cos 3\varphi \rightarrow 1$.

Example A.16 (Uniqueness fails for large k). For $n = 5$ and $m = 3$ — for instance the pair $(i, j) = (1, 4)$, which is not helical since $i + j = n$ — formula (39) gives $k^*(3, 5) = 22.4721 \dots$. For $k > k^*(3, 5)$ that path is p -independent and strictly shorter than D ; checking Lemma 8.3 shows it is also geometrically feasible for $k > 26.1335 \dots$. At $b = 1$, $k = 31$, $p = 0.25$,

$$d_1 = 31.81445 \dots, \quad d_3 = 31.98055 \dots, \quad D = 32,$$

so there are two distinct, feasible, non-helical p -independent paths shorter than the obvious one. An assertion of the form “there is exactly one such path” is therefore false without a restriction on k ; the correct statement is Theorem A.14. For $n = 3$ and $m = 2$ a second path appears already at $k > 6$.

A.8 The main result

Theorem A.17 (H1). *Let $n \geq 3$ and $k > 0$.*

- (a) *If n is odd, the regular n -prism carries a p -independent, geometrically feasible, non-helical path which for every $p \in [0, \frac{1}{2}]$ is strictly shorter than $D = b(k + 1)$. Its length is*

$$d = b \sqrt{k^2 + 2k \cos \frac{\pi}{n} + 1},$$

realised by the index pair $(\frac{n-1}{2}, \frac{n+1}{2})$ and its mirror.

- (b) *If n is even, no p -independent path — helical ones included — is shorter than the obvious path; each has length $\sqrt{(ms)^2 + D^2} \geq D$.*
- (c) *For $0 < k < k^*(n)$, where $k^*(n) \geq 6$, the path in (a) is the only p -independent path shorter than the obvious one.*

Proof. (a) is Theorem A.7; (b) is Theorem A.6; (c) is Theorem A.14. $\square$

Remark A.18 (Why odd prisms). Equation (37) shows that the only difference between the even and the odd case is the parameter $\varepsilon = b - 2r$. For even n the bisector joins the midpoints of two opposite sides, both at distance r from the centre, so $\varepsilon = 0$; then $\Delta y = D$ for *every* p -independent pair and shortening is impossible. For odd n the bisector joins a vertex (at distance R) to the midpoint of the opposite side (at distance r), so $\varepsilon = R - r > 0$; it is exactly this asymmetry that removes $\varepsilon(1 - \cos \theta)$ from the vertical component and makes $d < D$ possible. H1 is a direct consequence of the fact that a regular n -gon has a vertex-to-side bisector precisely when n is odd.

B H2: refuted, except at one point

H2 asserts that the number of shorter paths grows without bound as n increases while k stays bounded, helical paths excluded. We show that for every fixed $k \neq 1$ the number of classes is bounded — it saturates around $n \approx 100$ and does not move thereafter. H2 is therefore false. The exception is the single value $k = 1$, and it is exceptional for every p : there the number grows without bound, roughly like $\sqrt{n}$.

B.1 The numerical picture

Number of classes of shorter paths, counted with the exact criterion of Lemma 8.3 over pairs $0 \leq i, j \leq n$, at $p = 0.25$:

n	$k = 0.9$	$k = 1.0$	$k = 1.1$	$k = 1.5$	$k = 2$	$k = 3.5$	$k = 5$
201	7	257	115	33	21	12	9
801	7	407	114	29	20	10	9
1601	7	589	112	30	19	11	10
3201	7	849	109	29	19	11	9

Every column is flat except $k = 1$, which keeps growing. (Classes are formed by grouping lengths that agree to within a fixed tolerance; at these values of n many classes differ only in the tenth decimal or beyond, so the individual entries — unlike the saturation itself — are mildly sensitive to that tolerance.)

B.2 Why it saturates

Take $b = 1$, $\beta = 2\pi/n$, $x = i\beta$, $y = j\beta$, $w = \frac{1}{2} - p > 0$. As $n \rightarrow \infty$ we have $\sigma \rightarrow 0$ and $\rho \rightarrow \frac{1}{2}$, so the components of Theorem 8.1 become

$$\Delta x \rightarrow \frac{y - x}{2} - w(\sin x + \sin y), \quad \Delta y \rightarrow (k + 1) - w(\cos y - \cos x).$$

The feasibility criterion requires the two crossing points to fall inside the contact sides, which have width $\sigma = O(1/n)$. Both conditions therefore become *equalities* in the limit:

$$E_1 : (y - x)(1 - w(\cos y - \cos x)) + 2kw(\sin x + \sin y) = 0, \quad (41)$$

$$E_2 : \frac{1}{2}(\sin y - \sin x) + w \sin(x + y) = 0. \quad (42)$$

Two equations in two unknowns have isolated solutions. The lattice $(i\beta, j\beta)$ has spacing $\beta = 2\pi/n$ while the tolerance is $O(\sigma) = O(1/n)$ — the same order — so each solution accommodates $O(1)$ lattice points. The total count is therefore **bounded**, independently of n .

The prediction can be checked directly: at $n = 1601$, $k = 3.5$, $p = 0.25$ all 22 feasible shorter paths satisfy $|E_1|, |E_2| < 0.01 \approx \beta$, and they cluster around three isolated solutions.

B.3 Why $k = 1$ is different

The points $(0, 0)$ and (π, π) always solve the system. The Jacobian there is

$$\det J \Big|_{(0,0)} = \det J \Big|_{(\pi,\pi)} = -(k-1)(2p-1).$$

For $k \neq 1$ (and $p \neq \frac{1}{2}$) the intersection is transversal: an isolated point with $O(1)$ lattice points nearby. At $k = 1$ the determinant vanishes and the two curves $E_1 = 0$, $E_2 = 0$ become *tangent*. The region where both conditions hold to within $\varepsilon = O(1/n)$ is then not a patch of size $\varepsilon \times \varepsilon$ but an arc of length $\sim \sqrt{\varepsilon}$ and width $\sim \varepsilon$, containing

$$\frac{\sqrt{\varepsilon}}{\beta} \sim \sqrt{n}$$

lattice points.

The condition $k = 1$ does not involve p , which the measurements confirm: at $p = 0.1, 0.25$ and 0.4 the growth sets in at exactly $k = 1$. At $k = 1$, $p = 0.25$ the counts are 257, 313, 407, 589, 849 for $n = 201, 401, 801, 1601, 3201$, with local exponent rising towards 0.53 — consistent with $\sqrt{n}$.

Values near but not equal to 1 do saturate, though slowly: at $p = 0.4$, $k = 1.05$ the count rises to 2034 at $n = 1601$ and then falls to 1896 at $n = 3201$; at $k = 1.2$ it decreases steadily, $574 \rightarrow 560 \rightarrow 521 \rightarrow 502$.

B.4 Conclusion

- **H2 is false** as stated: for a typical fixed k the number of non-helical shorter paths does not grow without bound but saturates at a constant — 19 for $k = 2$, 11 for $k = 3.5$, 9 for $k = 5$ (at $p = 0.25$).
- It **holds exactly** at $k = 1$, for every $p \in [0, \frac{1}{2})$, where the number grows roughly like $\sqrt{n}$.
- A correct statement would be: *the number of non-helical shorter paths is bounded for every $k \neq 1$; at $k = 1$ it grows without bound.*

Caveat. Degeneracy has been established analytically only at $(0, 0)$ and (π, π) . It is not excluded that some other solution of the system becomes tangential at some other k , which would give a further exceptional value. A search over $k \in [0.8, 8]$ and $p \in \{0.1, 0.25, 0.4\}$ found none: everywhere except $k = 1$ the count saturates.

C H3: confirmed, with a growth law

H3 asserts that for fixed n the number of shorter paths grows without bound as $k \rightarrow \infty$, but only if helical paths are allowed. Both halves are true. In addition the growth turns out to be $C(n, p)\sqrt{k}$ with an explicit constant.

C.1 Structure

With $b = 1$, $u = p - \rho$, $v = p - 1 + \rho$ and $m = j - i$, Theorem 8.1 gives

$$\Delta x = m\sigma + X, \quad \Delta y = k + Y,$$

$$X = u \sin(c\beta) + v \sin(a\beta), \quad Y = 2\rho + u \cos(c\beta) - v \cos(a\beta),$$

where $a = i \bmod n$ and $c = j \bmod n$. The essential point is that X and Y depend *only* on the residues a, c , while m enters only linearly, in Δx . Helical paths are precisely those with large $|m|$, and for them X and Y are unchanged.

C.2 How many are shorter than the obvious path

From $d < D = k + 1$,

$$(m\sigma + X)^2 < (1 - Y)(2k + 1 + Y),$$

so $Y < 1$ is necessary. Given that, the admissible m form an interval of length

$$\frac{2\sqrt{(1 - Y)(2k + 1 + Y)}}{\sigma} \approx \frac{2\sqrt{2k(1 - Y)}}{\sigma},$$

which **grows like** $\sqrt{k}$. This is exactly where the restriction on helical paths matters: if $|m| \leq n$ the interval is truncated and nothing grows.

The condition $Y < 1$ simplifies unexpectedly. Writing $A = 1 - \cos(a\beta)$ and $C = 1 - \cos(c\beta)$, a direct computation gives $Y - 1 = (\rho - p)C - (1 - \rho - p)A$, so

$$(\rho - p)(1 - \cos(c\beta)) < (1 - \rho - p)(1 - \cos(a\beta)). \quad (43)$$

For even n , $\rho = \frac{1}{2}$ and this reduces to $\cos(c\beta) > \cos(a\beta)$. For $a = 0$ the right-hand side vanishes, so such a path is never shorter — that is the family of obvious paths.

C.3 Feasibility as $k \rightarrow \infty$

The exact criterion gives

$$x_P = \frac{\sigma}{2} - v \sin(a\beta) + \frac{\alpha}{\Delta y} \Delta x, \quad x_Q - m\sigma = \frac{\sigma}{2} + u \sin(c\beta) - \frac{\gamma}{\Delta y} \Delta x.$$

Since $\Delta x = O(\sqrt{k})$ while $\Delta y = O(k)$, the correction terms are $O(1/\sqrt{k})$ and vanish. In the limit, feasibility becomes a condition on the residues alone:

$$0 \leq \frac{\sigma}{2} - v \sin(a\beta) \leq \sigma, \quad 0 \leq \frac{\sigma}{2} + u \sin(c\beta) \leq \sigma. \quad (44)$$

This is the decisive contrast with Appendix B. There n grows, $\sigma \sim 1/n$ shrinks, and the conditions tighten into equations. Here n is fixed, σ does not shrink, and the conditions remain inequalities.

C.4 The growth law

For an admissible pair (a, c) only every n th value of m has the required residue, since $m \equiv c - a \pmod{n}$. Hence

Theorem C.1. *For fixed n and p , the number of shorter paths with helical paths admitted satisfies*

$$\#\{\text{paths}\} \sim C(n, p)\sqrt{k}, \quad C(n, p) = \frac{2}{n\sigma} \sum_{(a, c) \text{ admissible}} \sqrt{2(1 - Y(a, c))},$$

the sum being over residue pairs satisfying (43) and (44). The number of classes is half of this.

n	C (theory)	C at $k = 10^4$	C at $k = 5 \cdot 10^4$	deviation
3	3.6004	3.5800	3.5956	0.13 %
4	1.9142	1.9000	1.9141	0.01 %
5	6.7806	6.7800	6.7798	0.01 %
6	5.3650	5.3600	5.3845	0.36 %
7	7.5867	7.5400	7.6116	0.33 %
8	5.7880	5.7800	5.7869	0.02 %
9	5.3210	5.2800	5.3308	0.18 %

C.5 Measurements

Number of classes, helical paths admitted, $p = 0.25$:

k	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
5	4	2	9	4	12
20	9	3	13	7	12
100	18	7	35	17	32
500	40	14	76	38	82
2 000	80	27	151	75	170
10 000	179	60	339	169	377

The same, **without** helical paths ($0 \leq i, j \leq n$):

k	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
10	4	2	8	5	11
100	4	3	10	7	10
1 000	4	3	10	7	12
10 000	4	3	10	7	12
100 000	4	3	10	7	12

C.6 Conclusion

- **H3 is true.** With helical paths admitted the number of shorter paths grows without bound, for every $n \geq 3$.
- The growth is asymptotically $C(n, p)\sqrt{k}$, with the constant of Theorem C.1.
- The restriction to helical paths is essential, and for a clear reason: the saving comes from an interval of admissible m whose length grows like $\sqrt{k}$. Bound $|m|$ by n and the count saturates, as the second table shows.